WEEK II: FUNCTIONS, SEQUENCES

[Homework]

$1^{[2pt]}$. Determine if the following functions are injective/surjective/bijective. Write one proof.

$$\begin{aligned} f : \mathbb{R} \to \mathbb{R}, &\quad f(x) = x^3; \\ g : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{>0}, &\quad g(n) = n + 1; \\ h : \mathbb{R}^2 \to \mathbb{R}^2, &\quad h(x_1, x_2) = (x_1 - x_2, 2x_2 - 2x_1). \end{aligned}$$

$2^{[2pt]}$. Determine if the following real sequences are converging. Write one proof.

$$\begin{aligned} u_n &= (-1)^n; \\ v_n &= 1 + \frac{(-1)^n}{n}. \end{aligned}$$

$3^{[4pt]}$. In this question we look at the famous Fibonacci numbers using a matrix representation. Recall that the Fibonacci sequence $(F_n)_{n\geq 0}$ is defined by the recurrence relation:

$$F_n = F_{n-1} + F_{n-2}, \quad n > 1$$

with initial conditions $F_0 = 0$ and $F_1 = 1$.

(i) Calculate F_2, F_3, F_4, F_5 and F_6.

(ii) Prove that for all $n \geq 0$,

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{pmatrix} F_{n+2} \\ F_{n+1} \end{pmatrix}.$$

(iii) Diagonalize the matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

(iv) Calculate A^n for all $n \in \mathbb{Z}_{>0}$. Hint: you can use the question before (or not).

(v) Calculate the determinant $\det(A)$. Calculate $\det(A^n)$ for all $n > 0$.

(vi) Check that $A^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$ for all $n > 0$.

(vii) Prove the Cassini's identity $F_{n-1}F_{n+1} - (F_n)^2 = (-1)^n$ for all $n > 0$.

(viii) Give an explicit expression of F_n for all $n \geq 0$.

$4^{[2pt]}$. More on injectivity/surjectivity. Write two proofs of the following properties:
Let A, B, C be some sets and $f : A \to B$, $g : B \to C$ be two functions.

- If $g \circ f$ is injective then f is injective;
- If $g \circ f$ is surjective then g is surjective.

Let D be some non-empty set and $h : D \to D$ a function from D to itself.

- If $h \circ h = h$ then h is injective if and only if h is surjective;
- h is injective if and only if for all subsets $X \subset D$, $h^{-1}(h(X)) = X$;
- h is surjective if and only if for all subsets $X \subset D$, $h(h^{-1}(X)) = X$.

[Extra problem: subadditivity]

Let I be an interval in $\mathbb{R}$. We say that a function $f : I \to \mathbb{R}$ is subadditive if

$$\forall x, y \in I, \quad f(x+y) \leq f(x) + f(y).$$

Similarly we say that a real sequence $(u_n)_{n>0}$ is subadditive if

$$\forall n, m > 0, \quad u_{n+m} \leq u_n + u_m.$$

[Examples]

a) Show that the function $f : \mathbb{R}_{>0} \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$ is subadditive.
b) Show that any concave function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $f(0) \geq 0$ is subadditive.

[Fekete's subadditive lemma]

We prove the Fekete's lemma on subadditive sequences: if $(u_n)_{n>0}$ is a real subadditive sequence, then the limit of the sequence $(\frac{u_n}{n})_{n>0}$ exists in $\mathbb{R} \cup \{-\infty\}$ and we have

$$\lim_{n\to\infty} \frac{u_n}{n} = \inf_{n>0} \frac{u_n}{n} \in \mathbb{R} \cup \{-\infty\}.$$

c) Let $L = \inf_{n>0} \frac{u_n}{n} \in \mathbb{R} \cup \{-\infty\}$. Prove that $\forall \epsilon > 0$, $\exists n > 0$, $u_n \leq n(L+\epsilon)$.
d) Let $M = \max\limits_{1\leq i<n} u_i$. Prove that if $m > n, m = qn + r$ with $0 \leq r < n$, then $u_m \leq qu_n + M$.
e) Show that for large enough m, $\frac{u_m}{m} < L + \epsilon$ and conclude.

[Application: self avoiding walks]

We now give a famous application of the subadditivity to a real life and research problem.

Consider the grid $\mathbb{Z}^2$. Your favorite animal starts at the origin and moves either up, down, left, or right to the neighboring point at each step. Your animal cannot visit a point that it has already visited (this is called the "self-avoiding" property).

• How many different paths your animal can generate after $n > 0$ steps?
A) Let $c_n = c_n(\mathbb{Z}^2)$ be the above number. Verify that $c_1 = 4, c_2 = 12, c_3 = 36$ and calculate c_4.
B) Prove that $c_n \leq 4 \cdot 3^{n-1}$ for all $n > 0$.
C) By restricting your animal to only move up and right, prove that $c_n \geq 2^n$ for all $n > 0$.
D) Prove that $(c_n)_{n>0}$ is submultiplicative: $\forall n, m > 0$, $c_{n+m} \leq c_n \cdot c_m$.
E) Deduce that $\mu = \mu(\mathbb{Z}^2) = \lim\limits_{n\to\infty} c_n^{\frac{1}{n}}$ exists and $2 < \mu < 3$.

More information: the only known value of μ is for the hexagonal lattice $\mathbb{H}$ and $\mu(\mathbb{H}) = \sqrt{2+\sqrt{2}}$ (Hugo Duminil-Copin and Stanislav Smirnov, Annals of Mathematics, 2012). It is conjectured that for all lattices, $c_n \simeq \mu^n n^{\frac{11}{32}}$, where the constant $\frac{11}{32}$ does not depend on the lattice!

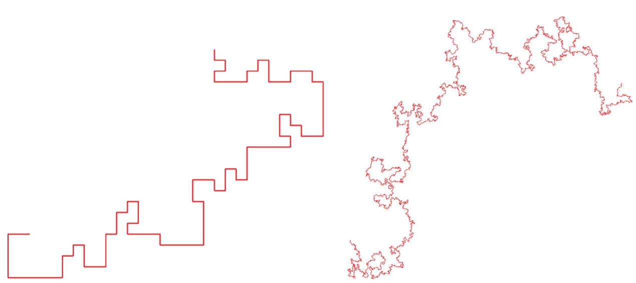

Figure 1. Examples of self-avoiding walks on $\mathbb{Z}^2$. ©Gordon Slade.

[Some hints]

1[2pt]. Determine if the following functions are injective/surjective/bijective. Write one proof.

$$\begin{aligned} f:\mathbb{R}\to\mathbb{R}, &\quad f(x)=x^3;\\ g:\mathbb{Z}_{\geq 0}\to\mathbb{Z}_{>0}, &\quad g(n)=n+1;\\ h:\mathbb{R}^2\to\mathbb{R}^2, &\quad h(x_1,x_2)=(x_1-x_2,2x_2-2x_1). \end{aligned}$$

• For the function f, it is a bijective function. A quick proof (that we will soon see) is that it is strictly increasing and unbounded (at both sides). An example of an ok proof is the following.

Let $y\in\mathbb{R}$ an element in the target set and solve the equation $f(x)=y$ with $x\in\mathbb{R}$ in the domain of definition of f. The equation writes $x^3=y$, and for every $y\in\mathbb{R}$, there exists a unique solution to the equation. Indeed, if $y\geq 0$, then $x=y^{\frac{1}{3}}$ is the unique solution; if $y<0$, then $x=-(-y)^{\frac{1}{3}}$ is the unique solution. By definition we have proven that f is a bijection between $\mathbb{R}$ and $\mathbb{R}$.

• For the function g, it is a bijective function. Injectivity follows from the fact that if $g(n)=n+1=g(m)=m+1$, then $n=m$. Surjectivity follows from the fact that for every $n>0$, $g(n-1)=n$ and $n-1\geq 0$ so that $n-1\in\mathbb{Z}_{\geq 0}$. This is an example of an ok proof.

• For the function h, it is not bijective. In fact, it is neither injective nor surjective. It is non injective since for example $h(1,1)=h(0,0)=(0,0)$ but $(0,0)\neq(1,1)$, and it is non surjective because if (y_1,y_2) is in the image of h, then necessarily $y_2=-2y_1$, so that for example $(0,1)$ is not in the image of h as $1\neq -2\cdot 0$.

2[2pt]. Determine if the following real sequences are converging. Write one proof.

$$\begin{aligned} u_n &= (-1)^n;\\ v_n &= 1+\frac{(-1)^n}{n}. \end{aligned}$$

• For the sequence u_n, it is not convergent since we can find two converging subsequences with different limits, one converging to 1 and the other one converging to -1. More precisely, let $v_n=u_{2n}$ and $w_n=u_{2n+1}$ by two subsequences of u_n. Since they are both constant sequences, $\lim\limits_{n\to\infty} v_n=1$ and $\lim\limits_{n\to\infty} w_n=-1$. This shows that u_n cannot be a converging sequence.

One can also use the $\epsilon-\delta$ definition. A rigorous proof can be written for example as follows: suppose that the sequence converges to $l\in\mathbb{R}$ and let $\epsilon=\frac{1}{2}$. Then there exists $n_0\in\mathbb{Z}_{>0}$, for all $N\geq n_0$, $|u_N-l|\leq\epsilon$. Apply this to $N+1$ and we get $|u_{N+1}-l|\leq\epsilon$, and by triangular inequality we have $|u_N-u_{N+1}|\leq 2\epsilon=1$. But $|u_N-u_{N+1}|=2$ by definition for all $N\in\mathbb{Z}_{>0}$, this yields a contradiction.

• For the sequence v_n, it converges to 1 (since for $\frac{(-1)^n}{n}$ we can look at its absolute value). An ok proof is as follows.

Since $-\frac{1}{n}\leq\frac{(-1)^n}{n}\leq\frac{1}{n}$ and $\lim\limits_{n\to\infty}\frac{1}{n}=0$, by squeeze theorem, $\lim\limits_{n\to\infty}\frac{(-1)^n}{n}=0$. Then by addition, $\lim\limits_{n\to\infty}\left(1+\frac{(-1)^n}{n}\right)=\lim\limits_{n\to\infty}1+\lim\limits_{n\to\infty}\frac{(-1)^n}{n}=1+0=1$.

3[4pt]. In this question we look at the famous Fibonacci numbers using a matrix representation.

Recall that the Fibonacci sequence $(F_n)_{n\geq 0}$ is defined by the recurrence relation:

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with initial conditions $F_0 = 0$ and $F_1 = 1$.

(i) Calculate F_2, F_3, F_4, F_5 and F_6.

(ii) Prove that for all $n \geq 0$,

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{pmatrix} F_{n+2} \\ F_{n+1} \end{pmatrix}.$$

(iii) Diagonalize the matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

(iv) Calculate A^n for all $n \in \mathbb{Z}_{>0}$. Hint: you can use the question before (or not).

(v) Calculate the determinant $\det(A)$. Calculate $\det(A^n)$ for all $n > 0$.

(vi) Check that $A^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$ for all $n > 0$.

(vii) Prove the Cassini's identity $F_{n-1}F_{n+1} - (F_n)^2 = (-1)^n$ for all $n > 0$.

(viii) Give an explicit expression of F_n for all $n \geq 0$.

i) $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$.

ii) Perform the matrix product and use the recurrence formula for the Fibonacci sequence.

iii) A quick way of finding the eigenvalues λ_1, λ_2 is to use that $\lambda_1 + \lambda_2 = \operatorname{tr}(A) = 1$ and $\lambda_1 \cdot \lambda_2 = \det(A) = -1$. One can take for example (thank you WolframAlpha) $D = \begin{pmatrix} \frac{1}{2}(1-\sqrt{5}) & 0 \\ 0 & \frac{1}{2}(1+\sqrt{5}) \end{pmatrix}$ and $P^{-1} = \begin{pmatrix} -\frac{1}{\sqrt{5}} & \frac{1}{10}(5+\sqrt{5}) \\ \frac{1}{\sqrt{5}} & \frac{1}{10}(5-\sqrt{5}) \end{pmatrix}$, $P = \begin{pmatrix} \frac{1}{2}(1-\sqrt{5}) & \frac{1}{2}(1+\sqrt{5}) \\ 1 & 1 \end{pmatrix}$.

iv) Use the fact that $P^{-1}AP = D$ implies $P^{-1}A^nP = D^n$.

v) $\det(A) = -1$. $\det(A^n) = \det(A)^n = (-1)^n$.

vi) This follows from a recurrence argument.

vii) Calculate the determinant of A^n using question v) and question vi).

viii) Use question vi) and question iv).

4[2pt]. More on injectivity/surjectivity. Write two proofs of the following properties:

Let A, B, C be some sets and $f : A \to B$, $g : B \to C$ be two functions.

- If $g \circ f$ is injective then f is injective;
- If $g \circ f$ is surjective then g is surjective.

Let D be some non-empty set and $h : D \to D$ a function from D to itself.

- If $h \circ h = h$ then h is injective if and only if h is surjective;
- h is injective if and only if for all subsets $X \subset D$, $h^{-1}(h(X)) = X$;
- h is surjective if and only if for all subsets $X \subset D$, $h(h^{-1}(X)) = X$.

This exercices is about manipulation of definitions.

• Suppose that $g \circ f$ is injective and prove that f is injective. Let $x, x' \in A$ such that $f(x) = f(x')$ and prove that $x = x'$. Since $f(x) = f(x')$, we have $g(f(x)) = g(f(x'))$. Injectivity of the function $g \circ f$ implies that $x = x'$, so that f is injective.

- Suppose that $g \circ f$ is surjective and prove that g is surjective. Let $y \in C$ and we will find $x \in B$ such that $g(x) = y$. Since $g \circ f : A \to C$ is surjective, we can find $t \in A$ such that $g(f(t)) = y$. Take $x = f(t)$, then $x \in B$ and $g(x) = y$. This shows that g is surjective.
- Suppose that $h \circ h = h$ and h is surjective, and we prove that h is also injective. Let $x, x' \in D$ such that $h(x) = h(x')$ and prove that $x = x'$. Since h is surjective, we can find $t, t' \in D$ such that $x = h(t)$ and $x' = h(t')$. It follows that $h(x) = h(h(t)) = h(t) = x$ and $h(x') = h(h(t')) = h(t') = x'$, so that $h(x) = h(x')$ implies $x = x'$: this shows that h is injective. For the other direction, suppose that h is injective and prove that h is also surjective. Let $y \in D$ and we will find $x \in D$ such that $y = h(x)$. Since $h \circ h = h$, we have that $h(y) = h(h(y))$. By injectivity of the function h, we have that $y = h(y)$. Take $x = y$ and we have $y = h(x)$: this means that h is surjective.
- Suppose that for all subsets $X \subset D$, $h^{-1}(h(X)) = X$ and prove that h is injective. Apply this to $X = \{x\} \subset D$ we get that $h^{-1}h(\{x\}) = x$, and by definition this means if $h(x') = h(x)$ then $x' = x$. Since this is true for all subset of type $\{x\} \subset D$, this shows that h is injective. For the other direction, suppose that h is injective and prove that for all subsets $X \subset D$, $h^{-1}(h(X)) = X$. Let $t \in h^{-1}(h(X))$ and prove that $t \in X$. By definition $h(t) \in h(X)$ so there exists $x \in X$ such that $h(t) = h(x)$. Since h is injective, $t = x$ so $h^{-1}(h(X)) \subset X$. The other inclusion is always true (try it yourself!).
- Suppose that for all subsets $X \subset D$, $h(h^{-1}(X)) = X$ and prove that h is surjective. Apply this to $X = \{y\} \subset D$ we get that $h^{-1}(X) \neq \emptyset$, so that h is surjective. To prove the other direction, suppose that h is surjective and take a subset $X \subset D$. Since h is surjective, for all $y \in X$ we can find $x \in D$ such that $h(x) = y$. This shows that $X \subset h(h^{-1}(X))$ and the other inclusion is always true.