Dependence Logic
Exercise 11

In these exercises, the letters $\mathsf{u}, \mathsf{v}, \mathsf{w}, \mathsf{x}, \mathsf{y}, \mathsf{z}, \dots$ in serif font stand for arbitrary (possibly empty) sequences of first-order variables.

Exercise 1. Verify that the rules $\subseteq \mathsf{Cmp}$, $\subseteq \mathsf{W}_\exists$ and $\subseteq \mathsf{W}_\forall$ in the system of inclusion logic FO($\subseteq$) are sound.

Exercise 2. Derive the following clauses in the system of inclusion logic:

(1) $\vdash \mathsf{x} \subseteq \mathsf{x}$ (In this exercise you are not allowed to use the rule $\subseteq \mathsf{Id}$).

(2) If $|\mathsf{x}| = |\mathsf{y}| = |\mathsf{z}|$, then $\mathsf{xy} \subseteq \mathsf{zz} \vdash \mathsf{x} = \mathsf{y}$, where $|\mathsf{x}|$ denotes the length of the sequence x.

(3) $\mathsf{xy} \subseteq \mathsf{uv} \vdash \mathsf{xyy} \subseteq \mathsf{uvv}$.

(4) $\vdash \forall x(y \subseteq x)$

Exercise 3. Verify that the rule $\perp_c \mathsf{I}$ in the system of independence logic FO($\perp$) is sound.

Exercise 4. Let $\tau = \{0, <\}$. Consider the τ-sentence

$$\phi = \forall xv \exists y\big(\big(v \geq 0 \to y < x + v\big) \land {=\!(x, y)}\big)$$

in FO($=\!(\ldots)$), where $v \geq 0 \to y < x + v$ is a shorthand for the formula $\neg(0 < v \lor 0 = v) \lor y < x + v$. Consider the game expression Φ of ϕ:

$$\begin{aligned}
\Phi =& \forall x_0 v_0 \exists y_0 \Big((v_0 \geq 0 \to y_0 < x_0 + v_0) \\
&\land \forall x_1 v_1 \exists y_1 \big((v_1 \geq 0 \to y_1 < x_1 + v_1) \land (x_1 = x_0 \to y_1 = y_0) \\
&\qquad \cdots \ \cdots \\
&\land \forall x_n v_n \exists y_n \big((v_n \geq 0 \to y_n < x_n + v_n) \land \bigwedge_{i<n} (x_n = x_i \to y_n = y_i) \land \ldots \\
&\qquad\qquad\qquad \cdots \ \cdots \qquad \big) \ldots \big) \Big).
\end{aligned}$$

(1) The sentence Φ is true on the model $\mathbb{Z} = (\mathbb{Z}, 0, <)$, where $\mathbb{Z}$ is the set of integers, the constant 0 is interpreted as the number zero, and $<$ is interpreted as the usual "less than" relation between integers. Describe a winning strategy for the player $\exists$loise in the infinite game $\mathcal{G}(\mathbb{Z}, \Phi)$.

(2) Does $(\mathbb{N}, 0, <) \models \phi$ hold? Justify your answer using the game $\mathcal{G}(\mathbb{N}, \Phi)$.

Exercise 5. Define a new type of atoms $\mathsf{x}\Upsilon\mathsf{y}$ as follows:

- $M \models_X \mathsf{x}\Upsilon\mathsf{y}$ iff for all $s \in X$, there exists $s' \in X$ such that $s(\mathsf{x}) = s'(\mathsf{x})$ and $s(\mathsf{y}) \neq s'(\mathsf{y})$.

These atoms are known as *afunctional dependencies* in database theory.

(1) Show that $\mathsf{x}\Upsilon\mathsf{y} \equiv \exists \mathsf{v}(\mathsf{xv} \subseteq \mathsf{xy} \wedge \mathsf{v} \neq \mathsf{y})$.

(2) The following clauses (when written as rules) are known to axiomatize completely the implication problem of afunctional dependencies:

(i) $\mathsf{xyz}\Upsilon\mathsf{uvw} \vdash \mathsf{yxz}\Upsilon\mathsf{uvw} \wedge \mathsf{xyz}\Upsilon\mathsf{vuw}$ (permutation).

(ii) $\mathsf{xy}\Upsilon\mathsf{z} \vdash \mathsf{x}\Upsilon\mathsf{zu}$ (monotonicity).

(iii) $\mathsf{xy}\Upsilon\mathsf{zy} \vdash \mathsf{xy}\Upsilon\mathsf{z}$ (weakening).

(iv) $\mathsf{x}\Upsilon\langle\,\rangle \vdash \bot$, where $\langle\,\rangle$ is the empty sequence.

Derive items (ii) and (iii) in the system of inclusion logic FO($\subseteq$) (where all of the atoms $\mathsf{x}\Upsilon\mathsf{y}$ are viewed as shorthands for their semantically equivalent formulas in the language of FO($\subseteq$), as given in item (1)) .