Dependence Logic

November 21 (Thursday), 2019

Normal Forms

- FO(=(...)):

$$\phi(\vec{v}) \vdash \forall \vec{x} \exists \vec{y} \Big(\bigwedge_{i \in I} =(\rho^i_{\vec{x}\vec{v}}, y_i) \wedge \alpha(\vec{x}, \vec{y}, \vec{v}) \Big)$$

- FO($\subseteq$):

$$\phi(\vec{v}) \vdash \exists \vec{x} \forall y \Big(\bigwedge_{i \in I} \rho^i_{\vec{x}\vec{v}y} \subseteq \sigma^i_{\vec{x}\vec{v}} \wedge \alpha(\vec{x}, y, \vec{v}) \Big)$$

- FO($\perp$):

$$\phi(\vec{v}) \vdash \forall \vec{x} \exists \vec{y} \Big(\bigwedge_{i \in I} \rho^i_{\vec{y}} \perp_{\pi^i_{\vec{y}}} \sigma^i_{\vec{y}} \wedge \alpha(\vec{x}, \vec{y}, \vec{v}) \Big)$$

Normal Forms

- FO(=(...)):

$$\phi(\vec{v}) \equiv \forall\vec{x}\exists\vec{y}\Big(\bigwedge_{i\in I} {=}(\rho^i_{\vec{x}\vec{v}}, y_i) \wedge \alpha(\vec{x},\vec{y},\vec{v})\Big)$$

- FO(⊆):

$$\phi(\vec{v}) \equiv \exists\vec{x}\forall y\Big(\bigwedge_{i\in I} \rho^i_{\vec{x}\vec{v}y} \subseteq \sigma^i_{\vec{x}\vec{v}} \wedge \alpha(\vec{x},y,\vec{v})\Big)$$

- FO(⊥):

$$\phi(\vec{v}) \equiv \forall\vec{x}\exists\vec{y}\Big(\bigwedge_{i\in I} \rho^i_{\vec{y}} \perp_{\pi^i_{\vec{y}}} \sigma^i_{\vec{y}} \wedge \alpha(\vec{x},\vec{y},\vec{v})\Big)$$

Normal Forms

- FO(=(...)):

$$\phi(\vec{v}) \vdash \forall \vec{x} \exists \vec{y} \Big(\bigwedge_{i \in I} {=}(\rho^i_{\vec{x}\vec{v}}, y_i) \wedge \alpha(\vec{x}, \vec{y}, \vec{v}) \Big)$$

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$$\phi(\vec{v}) \vdash \exists \vec{x} \forall y \Big(\bigwedge_{i \in I} \rho^i_{\vec{x}\vec{v}y} \subseteq \sigma^i_{\vec{x}\vec{v}} \wedge \alpha(\vec{x}, y, \vec{v}) \Big)$$

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- FO(=(...)):

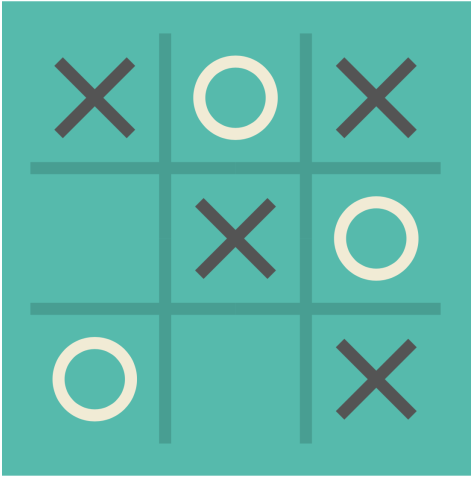
$$\phi \vdash \forall \vec{x} \exists \vec{y} \Big(\bigwedge_{i \in I} {=}(\rho^i_{\vec{x}}, y_i) \wedge \alpha(\vec{x}, \vec{y}) \Big)$$

- FO(⊆):

$$\phi \vdash \exists \vec{x} \forall y \Big(\bigwedge_{i \in I} \rho^i_{\vec{x}y} \subseteq \sigma^i_{\vec{x}} \wedge \alpha(\vec{x}) \Big)$$

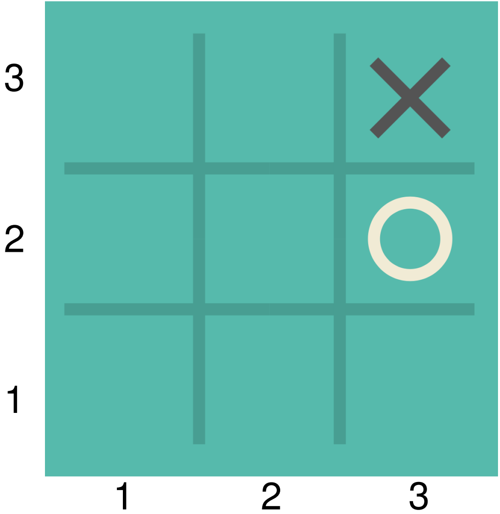
- FO(⊥):

$$\phi \vdash \forall \vec{x} \exists \vec{y} \Big(\bigwedge_{i \in I} \rho^i_{\vec{y}} \perp_{\pi^i_{\vec{y}}} \sigma^i_{\vec{y}} \wedge \alpha(\vec{x}, \vec{y}) \Big)$$



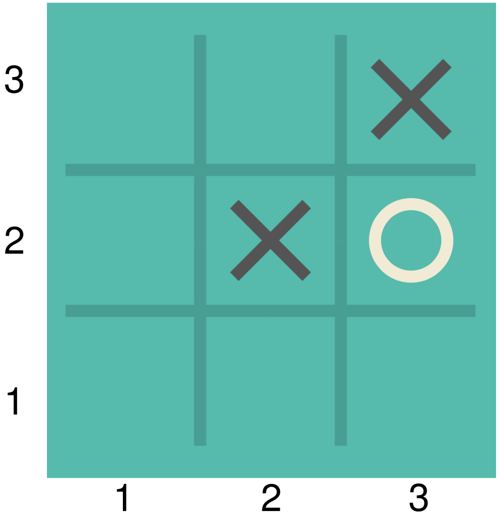
Abelard

Eloise



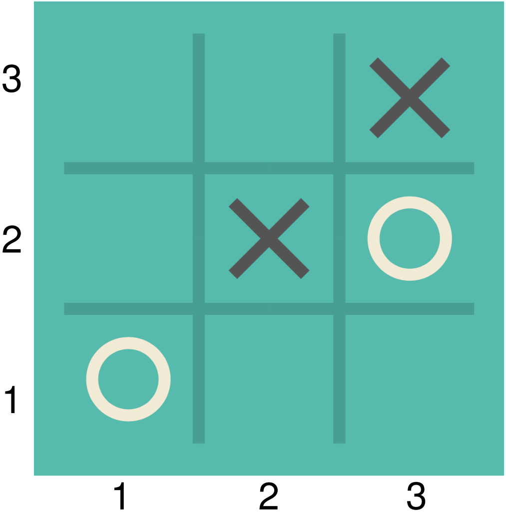
	Abelard	Eloise
round 1	(3,3)	(3,2)

Tic-tac-toe game



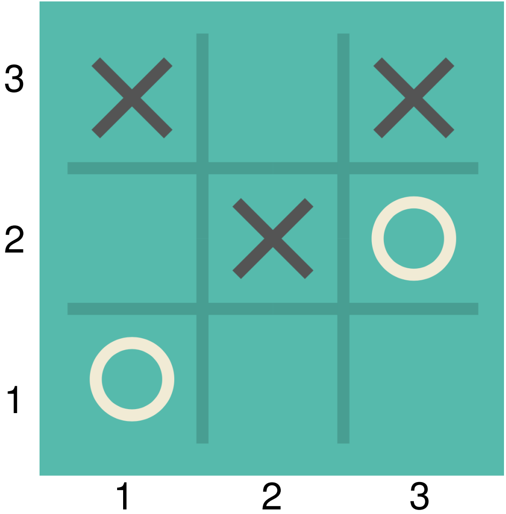
	Abelard	Eloise
round 1	(3,3)	(3,2)
round 2	(2,2)	

Tic-tac-toe game

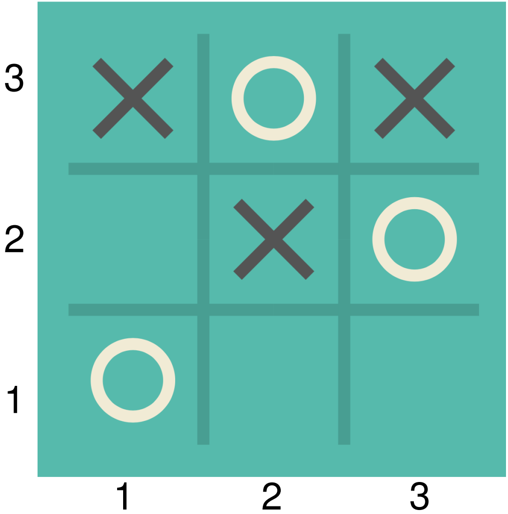



	Abelard	Eloise
round 1	(3,3)	(3,2)
round 2	(2,2)	(1,1)

Tic-tac-toe game

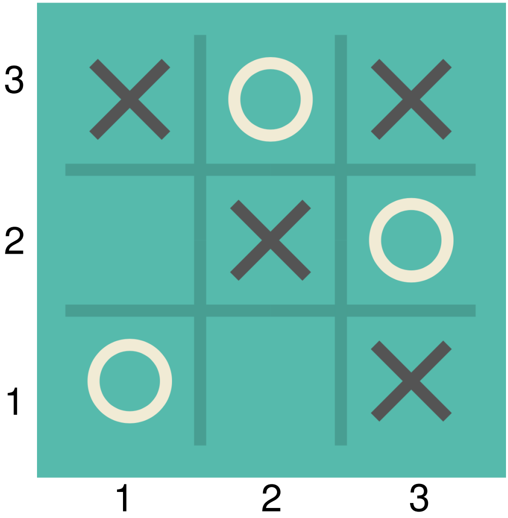


	Abelard	Eloise
round 1	(3,3)	(3,2)
round 2	(2,2)	(1,1)
round 3	(1,3)	



	Abelard	Eloise
round 1	(3,3)	(3,2)
round 2	(2,2)	(1,1)
round 3	(1,3)	(2,3)

Tic-tac-toe game



	Abelard	Eloise
round 1	(3,3)	(3,2)
round 2	(2,2)	(1,1)
round 3	(1,3)	(2,3)
round 4	(3,1)	–

Consider the following first-order sentence:

$$\begin{aligned}\Phi = \forall x_0 v_0 \exists y_0 \Big(& y_0 \le x_0 + v_0 \\ & \wedge \forall x_1 v_1 \exists y_1 \big(y_1 \le x_1 + v_1 \wedge (x_0 = x_1 \rightarrow y_0 = y_1) \\ & \quad \wedge \forall x_2 v_2 \exists y_2 (y_2 \le x_2 + v_2 \wedge (x_0 = x_2 \rightarrow y_0 = y_2) \wedge (x_1 = x_2 \rightarrow y_1 = y_2))\big)\Big)\end{aligned}$$

Game-theoretic semantics (simplified version)

Consider the following first-order sentence:

$$\Phi = \forall x_0 v_0 \exists y_0 \Big(y_0 \leq x_0 + v_0 \\ \land \forall x_1 v_1 \exists y_1 \big(y_1 \leq x_1 + v_1 \land (x_0 = x_1 \to y_0 = y_1) \\ \land \forall x_2 v_2 \exists y_2 (y_2 \leq x_2 + v_2 \land (x_0 = x_2 \to y_0 = y_2) \land (x_1 = x_2 \to y_1 = y_2))\big)\Big)$$

x_0	v_0	y_0
2	3	

Consider the following first-order sentence:

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$$\wedge \forall x_1 v_1 \exists y_1 \big(y_1 \leq x_1 + v_1 \wedge (x_0 = x_1 \rightarrow y_0 = y_1)$$
$$\wedge \forall x_2 v_2 \exists y_2 (y_2 \leq x_2 + v_2 \wedge (x_0 = x_2 \rightarrow y_0 = y_2) \wedge (x_1 = x_2 \rightarrow y_1 = y_2)) \big) \Big)$$

x_0	v_0	y_0
2	3	
		1

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Consider the following first-order sentence:

$$\Phi = \forall x_0 v_0 \exists y_0 \Big(y_0 \leq x_0 + v_0$$
$$\wedge \forall x_1 v_1 \exists y_1 \big(y_1 \leq x_1 + v_1 \wedge (x_0 = x_1 \rightarrow y_0 = y_1)$$
$$\wedge \forall x_2 v_2 \exists y_2 \big(y_2 \leq x_2 + v_2 \wedge (x_0 = x_2 \rightarrow y_0 = y_2) \wedge (x_1 = x_2 \rightarrow y_1 = y_2) \big) \big) \Big)$$

s

x_0	v_0	y_0
2	3	
		1

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$$\Phi = \forall x_0 v_0 \exists y_0 \Big(y_0 \leq x_0 + v_0$$
$$\wedge \forall x_1 v_1 \exists y_1 \big(y_1 \leq x_1 + v_1 \wedge (x_0 = x_1 \to y_0 = y_1)$$
$$\wedge \forall x_2 v_2 \exists y_2 (y_2 \leq x_2 + v_2 \wedge (x_0 = x_2 \to y_0 = y_2) \wedge (x_1 = x_2 \to y_1 = y_2)) \big) \Big)$$

s

x_0	v_0	y_0	x_1	v_1	y_1
2	3		4	2	
		1			

Game-theoretic semantics (simplified version)

Consider the following first-order sentence:

$$\Phi = \forall x_0 v_0 \exists y_0 \Big(y_0 \leq x_0 + v_0$$
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$$\wedge \forall x_2 v_2 \exists y_2 (y_2 \leq x_2 + v_2 \wedge (x_0 = x_2 \rightarrow y_0 = y_2) \wedge (x_1 = x_2 \rightarrow y_1 = y_2)) \big) \Big)$$

s

x_0	v_0	y_0	x_1	v_1	y_1
2	3		4	2	
		1			3

Game-theoretic semantics (simplified version)

Consider the following first-order sentence:

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$$\wedge \forall x_2 v_2 \exists y_2 \big(y_2 \leq x_2 + v_2 \wedge (x_0 = x_2 \rightarrow y_0 = y_2) \wedge (x_1 = x_2 \rightarrow y_1 = y_2) \big) \big) \Big)$$

s

x_0	v_0	y_0	x_1	v_1	y_1	x_2	v_2	y_2
2	3		4	2		2	8	
		1			3			1

Game-theoretic semantics (simplified version)

Consider the following first-order sentence:

$$\Phi = \forall x_0 v_0 \exists y_0 \Big(y_0 \leq x_0 + v_0$$
$$\wedge \forall x_1 v_1 \exists y_1 \big(y_1 \leq x_1 + v_1 \wedge (x_0 = x_1 \rightarrow y_0 = y_1)$$
$$\wedge \forall x_2 v_2 \exists y_2 \big(y_2 \leq x_2 + v_2 \wedge (x_0 = x_2 \rightarrow y_0 = y_2) \wedge (x_1 = x_2 \rightarrow y_1 = y_2) \big) \big) \Big)$$

$\mathcal{G}(\mathcal{M}, \Phi)$

s	x_0	v_0	y_0	x_1	v_1	y_1	x_2	v_2	y_2
∀belard	2	3		4	2		2	8	
∃loise			1			3			1

Game-theoretic semantics (simplified version)

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$$\begin{aligned}\Phi = \forall x_0 v_0 \exists y_0 \Big(& y_0 \leq x_0 + v_0 \\ & \wedge \forall x_1 v_1 \exists y_1 \big(y_1 \leq x_1 + v_1 \wedge (x_0 = x_1 \rightarrow y_0 = y_1) \\ & \wedge \forall x_2 v_2 \exists y_2 (y_2 \leq x_2 + v_2 \wedge (x_0 = x_2 \rightarrow y_0 = y_2) \wedge (x_1 = x_2 \rightarrow y_1 = y_2))\big)\Big)\end{aligned}$$

$\mathcal{G}(\mathcal{M}, \Phi)$

	round 0			round 1			round 2		
s	x_0	v_0	y_0	x_1	v_1	y_1	x_2	v_2	y_2
∀belard	2	3		4	2		2	8	
∃loise			1			3			1

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Consider the following first-order sentence:

$$\Phi = \forall x_0 v_0 \exists y_0 \Big(y_0 \leq x_0 + v_0$$
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$$\wedge \forall x_2 v_2 \exists y_2 \big(y_2 \leq x_2 + v_2 \wedge (x_0 = x_2 \rightarrow y_0 = y_2) \wedge (x_1 = x_2 \rightarrow y_1 = y_2) \big) \big) \Big)$$

$\mathcal{G}(\mathcal{M}, \Phi)$

	round 0			round 1			round 2		
s	x_0	v_0	y_0	x_1	v_1	y_1	x_2	v_2	y_2
∀belard	2	3		4	2		2	8	
∃loise			1			3			1

Game-theoretic semantics (simplified version)

Consider the following first-order sentence:

$$\Phi = \forall x_0 v_0 \exists y_0 \Big(y_0 \le x_0 + v_0$$
$$\wedge \forall x_1 v_1 \exists y_1 \big(y_1 \le x_1 + v_1 \wedge (x_0 = x_1 \to y_0 = y_1)$$
$$\wedge \forall x_2 v_2 \exists y_2 \big(y_2 \le x_2 + v_2 \wedge (x_0 = x_2 \to y_0 = y_2) \wedge (x_1 = x_2 \to y_1 = y_2) \big) \big) \Big)$$

$\mathcal{G}(\mathcal{M}, \Phi)$


	round 0			round 1			round 2		
s	x_0	v_0	y_0	x_1	v_1	y_1	x_2	v_2	y_2
∀belard	2	3		4	2		2	8	
∃loise			1			3			1

Consider the following first-order sentence:

$$\begin{aligned}\Phi = \forall x_0 v_0 \exists y_0 \Big(& y_0 \le x_0 + v_0 \\ & \wedge \forall x_1 v_1 \exists y_1 \big(y_1 \le x_1 + v_1 \wedge (x_0 = x_1 \to y_0 = y_1) \\ & \quad \wedge \forall x_2 v_2 \exists y_2 (y_2 \le x_2 + v_2 \wedge (x_0 = x_2 \to y_0 = y_2) \wedge (x_1 = x_2 \to y_1 = y_2)) \big) \Big)\end{aligned}$$

$\mathcal{G}(\mathcal{M}, \Phi)$

	round 0			round 1			round 2		
s	x_0	v_0	y_0	x_1	v_1	y_1	x_2	v_2	y_2
∀belard	2	3		4	2		2	8	
∃loise			1			3			1

Game-theoretic semantics (simplified version)

Consider the following first-order sentence:

$$\Phi = \forall x_0 v_0 \exists y_0 \Big(y_0 \leq x_0 + v_0$$
$$\wedge \forall x_1 v_1 \exists y_1 \big(y_1 \leq x_1 + v_1 \wedge (x_0 = x_1 \rightarrow y_0 = y_1)$$
$$\wedge \forall x_2 v_2 \exists y_2 \big(y_2 \leq x_2 + v_2 \wedge (x_0 = x_2 \rightarrow y_0 = y_2) \wedge (x_1 = x_2 \rightarrow y_1 = y_2) \big) \big) \Big)$$

$\mathcal{G}(\mathcal{M}, \Phi)$


	round 0			round 1			round 2		
s	x_0	v_0	y_0	x_1	v_1	y_1	x_2	v_2	y_2
∀belard	2	3		4	2		2	8	
∃loise			1			3			1

Def. ∃loise wins a play s of the game $\mathcal{G}(\mathcal{M}, \Phi)$

iff

$$\mathcal{M} \models_s \alpha(x_i, y_i) \wedge \gamma_i \text{ for all } i \leq 2.$$

Game-theoretic semantics (simplified version)

Consider the following first-order sentence:

$$\begin{aligned}\Phi = \forall x_0 v_0 \exists y_0 \Big(& y_0 \leq x_0 + v_0 \\ & \wedge \forall x_1 v_1 \exists y_1 \big(y_1 \leq x_1 + v_1 \wedge (x_0 = x_1 \rightarrow y_0 = y_1) \\ & \quad \wedge \forall x_2 v_2 \exists y_2 \big(y_2 \leq x_2 + v_2 \wedge (x_0 = x_2 \rightarrow y_0 = y_2) \wedge (x_1 = x_2 \rightarrow y_1 = y_2) \big) \big) \Big)\end{aligned}$$

$\mathcal{G}(\mathcal{M}, \Phi)$

	round 0			round 1			round 2		
s	x_0	v_0	y_0	x_1	v_1	y_1	x_2	v_2	y_2
∀belard	2	3		4	2		2	8	
∃loise			1			3			1

Def. ∃loise **wins** a play s of the game $\mathcal{G}(\mathcal{M}, \Phi)$

iff

$\mathcal{M} \models_s \alpha(x_i, y_i) \wedge \gamma_i$ for all $i \leq 2$.

Thm. $\mathcal{M} \models \Phi$ iff ∃loise has a **winning strategy** in the game $\mathcal{G}(\mathcal{M}, \Phi)$.

Game-theoretic semantics (simplified version)

Consider the following first-order sentence:

$$\begin{aligned}\Phi^n = \forall \vec{x}_0 \exists \vec{y}_0 \Big(\alpha(\vec{x}_0, \vec{y}_0) \\ \wedge \forall \vec{x}_1 \exists \vec{y}_1 \big(\alpha(\vec{x}_1, \vec{y}_1) \wedge \gamma_1 \\ \cdots \\ \wedge \forall \vec{x}_n \exists \vec{y}_n \big(\alpha(\vec{x}_n, \vec{y}_n) \wedge \gamma_n \big) \big) \Big)\end{aligned}$$

$\mathcal{G}(\mathcal{M}, \Phi^n)$


	round 0		round 1		$\cdots$	round n	
s	$\vec{x}_0$	$\vec{y}_0$	$\vec{x}_1$	$\vec{y}_1$	$\ldots$	$\vec{x}_n$	$\vec{y}_n$
$\forall$belard	$\vec{a}_0$		$\vec{a}_1$			$\vec{a}_n$	
$\exists$loise		$\vec{b}_0$		$\vec{b}_1$			$\vec{b}_n$

Def. $\exists$loise wins a play s of the game $\mathcal{G}(\mathcal{M}, \Phi^n)$

iff

$$\mathcal{M} \models_s \alpha(x_i, y_i) \wedge \gamma_i \text{ for all } i \leq n.$$

Thm. $\mathcal{M} \models \Phi^n$ iff $\exists$loise has a winning strategy in the game $\mathcal{G}(\mathcal{M}, \Phi^n)$.

Infinite formulas and infinite games (simplified version)

Consider the following infnitary first-order sentence:

$$\Phi = \forall \vec{x_0} \exists \vec{y_0} \Big(\alpha(\vec{x_0}, \vec{y_0}) \wedge \forall \vec{x_1} \exists \vec{y_1} \big(\alpha(\vec{x_1}, \vec{y_1}) \wedge \gamma_1 \cdots \wedge \forall \vec{x_n} \exists \vec{y_n} \big(\alpha(\vec{x_n}, \vec{y_n}) \wedge \gamma_n \wedge \ldots \ldots \ldots \quad \ldots \big) \ldots \big) \Big)$$

$\mathcal{G}(\mathcal{M}, \Phi)$


	round 0		round 1		$\cdots$	round n		
s	$\vec{x_0}$	$\vec{y_0}$	$\vec{x_1}$	$\vec{y_1}$	$\ldots$	$\vec{x_n}$	$\vec{y_n}$	$\ldots$
∀belard	$\vec{a_0}$		$\vec{a_1}$			$\vec{a_n}$		$\ldots$
∃loise		$\vec{b_0}$		$\vec{b_1}$			$\vec{b_n}$	

Def. ∃loise wins a play s of the (infinite) game $\mathcal{G}(\mathcal{M}, \Phi)$
iff
$\mathcal{M} \models_s \alpha(x_n, y_n) \wedge \gamma_n$ for all $n \in \mathbb{N}$.

Thm. $\mathcal{M} \models \Phi$ iff ∃loise has a winning strategy in the game $\mathcal{G}(\mathcal{M}, \Phi)$.

$$\begin{aligned}\Phi = \forall x_0 v_0 \exists y_0 \Big(& y_0 \leq x_0 + v_0 \\ & \wedge \forall x_1 v_1 \exists y_1 \big(y_1 \leq x_1 + v_1 \wedge (x_0 = x_1 \rightarrow y_0 = y_1) \\ & \wedge \forall x_2 v_2 \exists y_2 (y_2 \leq x_2 + v_2 \wedge (x_0 = x_2 \rightarrow y_0 = y_2) \wedge (x_1 = x_2 \rightarrow y_1 = y_2)) \big) \Big)\end{aligned}$$

$\mathcal{G}(\mathcal{M}, \Phi)$


	round 0			round 1			round 2		
s	x_0	v_0	y_0	x_1	v_1	y_1	x_2	v_2	y_2
∀belard	2	3		4	2		2	8	
∃loise			1			3			1

$$\Phi = \forall x_0 v_0 \exists y_0 \Big(y_0 \leq x_0 + v_0$$
$$\wedge \forall x_1 v_1 \exists y_1 \big(y_1 \leq x_1 + v_1 \wedge (x_0 = x_1 \rightarrow y_0 = y_1)$$
$$\wedge \forall x_2 v_2 \exists y_2 \big(y_2 \leq x_2 + v_2 \wedge (x_0 = x_2 \rightarrow y_0 = y_2) \wedge (x_1 = x_2 \rightarrow y_1 = y_2) \big)\big)\Big)$$

$\mathcal{G}(\mathcal{M}, \Phi)$

	round 0	round 1	round 2
s	x_0 v_0 y_0	x_1 v_1 y_1	x_2 v_2 y_2
∀belard	2 3	4 2	2 8
∃loise	1	3	1

∀belard ∃loise

	x v	y
round 0	2 3	1
round 1	4 2	3
round 2	2 8	1

$$\Phi = \forall x_0 v_0 \exists y_0 \Big(y_0 \leq x_0 + v_0$$
$$\wedge \forall x_1 v_1 \exists y_1 \big(y_1 \leq x_1 + v_1 \wedge (x_0 = x_1 \rightarrow y_0 = y_1)$$
$$\wedge \forall x_2 v_2 \exists y_2 \big(y_2 \leq x_2 + v_2 \wedge (x_0 = x_2 \rightarrow y_0 = y_2) \wedge (x_1 = x_2 \rightarrow y_1 = y_2) \big) \big) \Big)$$

$\mathcal{G}(\mathcal{M}, \Phi)$

	round 0			round 1			round 2		
s	x_0	v_0	y_0	x_1	v_1	y_1	x_2	v_2	y_2
∀belard	2	3		4	2		2	8	
∃loise			1			3			1

	∀belard	∃loise
	x v	y
s_0	2 3	1
s_1	4 2	3
s_2	2 8	1

Games and teams ... and dependence

$$\Phi = \forall x_0 v_0 \exists y_0 \Big(y_0 \leq x_0 + v_0$$
$$\wedge \forall x_1 v_1 \exists y_1 \big(y_1 \leq x_1 + v_1 \wedge (x_0 = x_1 \to y_0 = y_1)$$
$$\wedge \forall x_2 v_2 \exists y_2 \big(y_2 \leq x_2 + v_2 \wedge (x_0 = x_2 \to y_0 = y_2) \wedge (x_1 = x_2 \to y_1 = y_2)\big)\big)\Big)$$

	$\forall$	$\exists$
	x v	y
s_0	2 3	1
s_1	4 2	3
s_2	2 8	1

$$\phi = \forall xv \exists y (y \leq x + v \wedge {=}(x, y))$$

Games and teams ... and dependence

$$\Phi = \forall x_0 v_0 \exists y_0 \Big(y_0 \leq x_0 + v_0$$
$$\wedge \forall x_1 v_1 \exists y_1 \big(y_1 \leq x_1 + v_1 \wedge (x_0 = x_1 \to y_0 = y_1)$$
$$\wedge \forall x_2 v_2 \exists y_2 \big(y_2 \leq x_2 + v_2 \wedge (x_0 = x_2 \to y_0 = y_2) \wedge (x_1 = x_2 \to y_1 = y_2)$$
$$\wedge \forall x_3 v_3 \exists y_3 \big(y_3 \leq x_3 + v_3 \wedge \textstyle\bigwedge_{i<3} (x_i = x_3 \to y_i = y_3)\big)\big)\big)\Big)$$

	$\forall$	$\exists$
	x v	y
s_0	2 3	1
s_1	4 2	3
s_2	2 8	1
s_3	4 5	3

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Games and teams ... and dependence

$$\Phi = \forall x_0 v_0 \exists y_0 \Big(y_0 \leq x_0 + v_0$$
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$$\dots \dots \quad) \dots)) \Big)$$

	$\forall$	$\exists$
	$x\ v$	y
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⋮	⋮	

$$\phi = \forall xv \exists y (y \leq x + v \wedge {=}(x, y))$$

Idea: The infinitary FO-sentence Φ simulates the FO(=(...))-sentence ϕ.

The game expression

Let $\phi = \forall\vec{x}\exists\vec{y}\Big(\alpha(\vec{x},\vec{y}) \wedge \bigwedge_{k\in K} =\!(\rho^k_{\vec{x}}, y_k)\Big)$ be a sentence of FO(=(...)).

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Thm. If $\mathcal{M}$ is countable, then $\mathcal{M}\models\phi \iff \mathcal{M}\models\Phi$.

For each $n \in \mathbb{N}$, define the *n-approximation* Φ^n of Φ as the (finite) first-order sentence

$$\begin{aligned}\Phi^n = & \forall \vec{x_0} \exists \vec{y_0} \Big(\alpha(\vec{x_0}, \vec{y_0}) \\ & \quad \wedge \forall \vec{x_1} \exists \vec{y_1} \big(\alpha(\vec{x_1}, \vec{y_1}) \wedge \gamma_1 \\ & \qquad \ldots \ldots \\ & \qquad \wedge \forall \vec{x_n} \exists \vec{y_n} \big(\alpha(\vec{x_n}, \vec{y_n}) \wedge \gamma_n \big) \big) \Big).\end{aligned}$$

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& \qquad \dots \ \dots \\
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Thm. If $\mathcal{M}$ is a recursively saturated (or finite) model, then

$$\mathcal{M} \models \Phi \iff \mathcal{M} \models \Phi^n \text{ for all } n \in \mathbb{N}.$$

In particular, if $\mathcal{M}$ is a countable recursively saturated (or finite) model, then

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A model $\mathcal{M}$ is called *recursively saturated*, if for any recursive set $\{\psi_n(\mathrm{x}, \mathrm{y}) \mid n < \omega\}$ of formulas,

$$\mathcal{M} \models \forall \mathrm{x} \Big(\bigwedge_{n<\omega} \exists \mathrm{y} \bigwedge_{m \leq n} \psi_m(\mathrm{x}, \mathrm{y}) \to \exists \mathrm{y} \bigwedge_{n<\omega} \psi_n(\mathrm{x}, \mathrm{y}) \Big).$$

For every infinite model $\mathcal{M}$, there is a recursively saturated countable model $\mathcal{M}'$ s.t. $\mathcal{M} \equiv \mathcal{M}'$

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- Trivially, $\phi \vdash \Psi^0$ holds, as Ψ^0 is ϕ (modulo renaming of variables).
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  Then $\Psi^n \vdash \Psi^{n+1}$ follows from monotonicity. □

Dependence atom elimination rule

$$\frac{\forall\vec{x}\exists\vec{y}\Big(\bigwedge_{i\in I}{=}(\rho^i_{\vec{x}\vec{y}\vec{v}}, y_i) \wedge \phi(\vec{x},\vec{y},\vec{v})\Big)}{\forall\vec{x}\exists\vec{y}\Big(\phi(\vec{x},\vec{y},\vec{v})\wedge \forall\vec{x}'\exists\vec{y}'\big(\phi(\vec{x}',\vec{y}',\vec{v})\wedge\bigwedge_{i\in I}(\rho^i_{\vec{x}\vec{y}\vec{v}}=\rho^i_{\vec{x}'\vec{y}'\vec{v}}\rightarrow y_i=y'_i)\big)\Big)}\ {=}(\,)\mathrm{E}$$

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