Dependence Logic

Exercise 10

Exercise 1. Let $\alpha(x_1, \ldots, x_n)$ be a first-order τ-formula, and R be a relation symbol that is not in τ. Show that for any τ-model $\mathcal{M}$ and team X,

$$\mathcal{M} \models_X \alpha(\vec{x}) \iff (\mathcal{M}, X[\vec{x}]) \models \forall x(R\vec{x} \to \alpha(\vec{x})),$$

where $X[\vec{x}] = \{(s(x_1), \ldots, s(x_n)) \mid s \in X\}$ is the interpretation of the new relation symbol R.

Exercise 2. Consider the connective $\to$ defined as follows:

$$\mathcal{M} \models_X \phi \to \psi \iff \text{for all } Y \subseteq X, \mathcal{M} \models_Y \phi \text{ implies } \mathcal{M} \models_Y \psi.$$

Prove the following:

(1) For any set $\Gamma \cup \{\phi, \psi\}$ of formulas in dependence logic FO(=(...)),

$$\Gamma, \phi \models \psi \iff \Gamma \models \phi \to \psi.$$

(2) $=\!(x_1, \ldots, x_n, y) \equiv \big(=\!(x_1) \wedge \cdots \wedge =\!(x_n)\big) \to =\!(y)$.

(3) Conclude from item (2) that the two rules $=(\,)\mathsf{I}_0$ and $=(\,)\mathsf{E}_0$ in the system of dependence logic FO(=(...)) are sound.

Exercise 3. Verify that the rule $=(\,)_\vee\mathsf{Ext}$ in the system of dependence logic is sound.

Exercise 4. Derive the following clauses in the system of dependence logic:

(1) If $x \notin \mathsf{Fv}(\psi)$, then $\forall x\phi \vee \psi \vdash \forall x(\phi \vee \psi)$.

(2) $\vdash \exists y =\!(x, y)$ and $\vdash \exists x =\!(x)$ (Hint: Introduce dummy variables).

(3) $\forall\vec{x}\exists y\phi(\vec{x}, y, \vec{v}) \vdash \forall\vec{x}\exists y(\phi \wedge =\!(\vec{x}\vec{v}, y))$.

(4) $=\!(\vec{x}, y), =\!(\vec{x}y\vec{v}, z) \vdash =\!(\vec{x}\vec{v}, z)$.

Exercise 5. (1) Prove Lemma 3.5.12 (Substitution lemma) in the lecture notes.

(2) Show that the rule $\forall\mathsf{E}$ in the system of dependence logic is sound.