WEEK IV: SOME EXAMPLES OF PROOFS

1. Check definitions

1.1. **Something has property (P).** One should check carefully item by item that the property (P) is satisfied by the object we consider.

Example 1 (Distance)**.** Let X be a set. A distance (or a metric) on X is an application $d : X \times X \to \mathbb{R}_{>0}$ such that for all $x, y, z \in X$, the following conditions are satisfied:

(1) Non-negativity or separation:
$$d(x, y) \geq 0;$$
(2) Identity of indiscernibles:
$$d(x, y) = 0 \iff x = y;$$
(3) Symmetry:
$$d(x, y) = d(y, x);$$
(4) Subadditivity or triangular inequality:
$$d(x, y) \leq d(x, z) + d(z, y).$$

Show that the discrete metric on X such that $d(x, x) = 0$ and $d(x, y) = 1$ if $x \neq y$ is a distance.

Proof. Let d be the discrete metric on X. Let $x, y, z \in X$.

• Separation: if $x = y$ then $d(x, y) = 0 \geq 0$; if $x \neq y$ then $d(x, y) = 1 \geq 0$. Therefore the function d is non-negative.

• Identity of indiscernibles: if $x = y$ then $d(x, y) = 0$. Conversely, we prove the contrapositive: if $x \neq y$ then $d(x, y) \neq 0$. Indeed, if $x \neq y$ then $d(x, y) = 1 \neq 0$. Together we have $d(x, y) = 0 \iff x = y$.

• Symmetry: it suffices to consider the case where $x \neq y$. If $x \neq y$, $d(x, y) = 1$ and $d(y, x) = 1$ so $d(x, y) = d(y, x)$. The function d is thus symmetric.

• Triangular inequality: if $x = y$ then $d(x, y) = 0$ and $d(x, z) + d(z, y) \geq 0$ by positivity of the distance so $d(x, y) \leq d(x, z) + d(z, y)$. If $x \neq y$, then either $z \neq x$ or $z \neq y$. In this case, $d(x, y) = 1$ and $d(x, z) + d(z, y) \geq 1$ so $d(x, y) \leq d(x, z) + d(z, y)$.

In conclusion, d defines a distance on X. □

Exercise 1 (Inclusion is a partial order)**.** A partial order is a binary relation[1] $\leq$ over a set P satisfying the following (reflexive, antisymmetric, transitive) axioms for all $a, b, c \in P$:

(1) Reflexivity: $a \leq a$;
(2) Antisymmetry: if $a \leq b$ and $b \leq a$ then $a = b$;
(3) Transitivity: if $a \leq b$ and $b \leq c$ then $a \leq c$.

Let A be any set and $\mathcal{P}(A)$ be the set of its subsets: $\mathcal{P}(A) = \{X; X \subset A\}$. Prove that the inclusion $\subset$ is a partial order on $\mathcal{P}(A)$.

[1] A binary relation on P is a set of <u>ordered</u> pairs (p, q) with $p, q \in P$: we write $p \leq q$ if (p, q) belongs to the set.

1.2. **Something does not have property (P).** One has to know how to express the negation of a property (P).

Example 2 (Intermediate value property does not imply continuity)**.** Show that the function $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = \sin(\frac{1}{x})$ for $0 < x \leq 1$ and $f(0) = 0$ satisfies the intermediate value property but is not continuous at 0.

Proof. First we prove that f satisfies the intermediate value property, then we prove the discontinuity of f at 0.

• Intermediate value property: consider $a < b$ and let $[a, b] \subset [0, 1]$ be a closed interval. If $a \neq 0$, then f is continuous on $[a, b]$ by composition of continuous functions so that f has the intermediate value property on $[a, b]$. Suppose then $a = 0$ and consider the interval $[0, b] \subset [0, 1]$ with $b \neq 0$, and prove that f takes all values between $[f(0), f(b)] = [0, f(b)]$. It is enough to show that f takes all values between $[-1, 1]$ on the interval $[0, b]$. Indeed, if we can find two points $x, y \in (0, b)$ such that $f(x) = 1$ and $f(y) = -1$ then by the continuity of f on $[x, y] \subset (0, 1]$, f has the intermediate value property on the closed interval between x and y, in such a way that f reaches all values between $[-1, 1]$ on the interval delimited by x and y: as a consequence f reaches all values between $[-1, 1]$ on $[0, b]$. One verifies that the following choices work: $x^{-1} = 2\pi\lfloor\frac{b}{2\pi}\rfloor + \frac{5\pi}{2}$ and $y^{-1} = 2\pi\lfloor\frac{b}{2\pi}\rfloor + \frac{7\pi}{2}$.

• Discontinuity at 0: we prove that the limit of $f(x)$ as x approaches 0^+ does not exists. Since $f(0) = 0$, this limit if it exists must be equal to 0: we prove that this is not the case. Take $\epsilon = \frac{1}{2}$ and prove that for all $\delta > 0$, there exists $x \in]-\delta, \delta[$ such that $|f(x) - f(0)| = |f(x)| \geq \frac{1}{2}$. We have proven above that for all $\delta > 0$, the function f reaches all values in between $[-1, 1]$ on the interval $[0, \delta]$: in particular there exists $x \in [0, \delta]$ such that $|f(x)| = 1 > \frac{1}{2}$. The function f is thus not continuous at 0. □

Exercise 2 (Limit of a function)**.** Show that the function $g : [0, 1] \to \mathbb{R}$ defined by $g(x) = x$ if $0 < x \leq 1$ and $g(0) = 1$ is not continuous at 0.

Suppose that $\lim_{x \to 0} g(x)$ exists. Since $0 \in [0, 1]$ is in the domain of definition of g and $g(0) = 1$, we must have $\lim_{x \to 0} g(x) = 1$.

Let $\epsilon = \frac{1}{2}$ and prove that we cannot find $\delta > 0$ such that for all $x \in [0, \delta)$, $|g(x) - g(0)| = |g(x) - 1| < \epsilon = \frac{1}{2}$. Indeed, suppose such a $\delta > 0$ exists and consider $x = \min(\frac{\delta}{2}, \frac{1}{4}) \in (0, \delta)$. By definition $0 < x \leq \frac{1}{4}$ and so that $|g(x) - 1| \geq \frac{3}{4} > \frac{1}{2} = \epsilon$. This is a contradiction, thus such δ does not exist and the function g is not continuous at 0.

Another way of proving this is to show that g does not verify the intermediate value property for example on the closed interval $[0, \frac{1}{4}]$.

2. Sets

2.1. **X is equal to Y.** The general strategy for proving $X = Y$ is to subdivide the question into two parts: show that $X \subset Y$ and also that $Y \subset X$. The general strategy for proving $X \subset Y$ is to take an element $x \in X$ and prove that $x \in Y$.

Example 3 ($P \implies Q$ is the same as $\neg P \vee Q$)**.** Let P, Q two subsets of a set A. Prove that

$$P \subset Q \iff (A \backslash P) \cup Q = A.$$

Proof. We will prove the two implications.

$\implies$: Let $P \subset Q$ and prove that $(A\backslash P) \cup Q = A$. We prove first that $(A\backslash P) \cup Q \subset A$. This is because $A\backslash P \subset A$ and $Q \subset A$, so is their union[2]. We now prove that $A \subset (A\backslash P) \cup Q$ assuming $P \subset Q$. Let $a \in A$ and prove that $a \in (A\backslash P) \cup Q$. If $a \in Q$ then we are done, so suppose that $a \notin Q$. Since $P \subset Q$ and $a \notin Q$, we have in particular $a \notin P$. Since $a \in A$ and $a \notin P$, we have $a \in A\backslash P$, so $a \in (A\backslash P) \cup Q$ in all cases.

$\Longleftarrow$: Suppose that $(A\backslash P) \cup Q = A$ and prove that $P \subset Q$. Let $p \in P$ and prove that $p \in Q$. Since $p \in P$, we have that $p \in A$ so that $p \in (A\backslash P) \cup Q$. Since $p \in P$, $p \notin A\backslash P$ so that necessarily $p \in Q$. □

Exercise 3 (Preimage has better properties than image)**.** Let E, F be two non-empty sets and $f : E \to F$ a function. Recall that if $\mathcal{P}(X)$ stands for the set of subsets of a set X (see the previous exercise) then f also defines two applications:

(Image) $$f : \mathcal{P}(E) \to \mathcal{P}(F); \quad E \supset A \mapsto f(A) = \{f(a); a \in A\} \subset F.$$

(Preimage) $$f^{-1} : \mathcal{P}(F) \to \mathcal{P}(E); \quad F \supset C \mapsto f^{-1}(C) = \{e \in E; f(e) \in C\} \subset E.$$

Prove the following statements:

(1) $\forall A, B \subset E$, $f(A \cup B) = f(A) \cup f(B)$;
(2) $\forall A, B \subset E$, $f(A \cap B) \subset f(A) \cap f(B)$;
(3) $\forall C, D \subset F$, $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$;
(4) $\forall C, D \subset F$, $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.

Also, give an example where for the second statement, $f(A \cap B) \neq f(A) \cap f(B)$.

3. Implications

3.1. **(A) is equivalent to (B).** One shows that (A) implies (B) and then (B) implies (A).

Example 4 (Equivalent characterizations of set inclusion)**.** For any two subsets A and B of some set X, the following are equivalent:

(1) $A \subset B$;
(2) $A \cap B = A$;
(3) $A \cup B = B$;
(4) $A\backslash B = \emptyset$;
(5) $X\backslash B \subset X\backslash A$.

Proof. We show that (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) $\implies$ (5) $\implies$ (1) (why is it enough?).

• (1) $\implies$ (2): Suppose that $A \subset B$ and prove that $A \cap B = A$. First, $A \cap B \subset A$ by definition. To prove that $A \subset A \cap B$ when $A \subset B$, let $a \in A$ and prove that $a \in A \cap B$. This reduces to show that $a \in B$, but this is true since $a \in A$ and $A \subset B$.

• (2) $\implies$ (3): Suppose that $A \cap B = A$ and prove that $A \cup B = B$. First, $B \subset A \cup B$ by definition. To prove that $A \cup B \subset B$ when $A \cap B = A$, let $a \in A \cup B$ and show that $a \in B$. Since $a \in A \cup B$, then $a \in B$ (in which case we are done) or $a \in A$. If $a \in A$ then by assumption $a \in A \cap B$ and since $A \cap B \subset B$, we have $a \in B$.

• (3) $\implies$ (4): Suppose that $A \cup B = B$ and prove that $A\backslash B = \emptyset$. Since $\emptyset \subset A\backslash B$ is always true, we prove that there is no element in $A\backslash B$. By contradiction, suppose that $a \in A\backslash B$, then $a \in A$ and $a \notin B$. Since $A \subset A \cup B$, we have that $a \in A \cup B = B$ and $a \notin B$, contradiction. So $A\backslash B = \emptyset$.

• (4) $\implies$ (5): Suppose that $A\backslash B = \emptyset$ and prove that $X\backslash B \subset X\backslash A$. Let $x \in X\backslash B$ and prove that $x \in X\backslash A$. We have that $x \in X$ and $x \notin B$, and it suffices to prove that $x \notin A$. By

[2] If one is unsure, verify the following statement: if $X \subset A$ and $Y \subset A$, then $X \cup Y \subset A$ and $X \cap Y \subset A$. This part of the proof does not use the assumption $P \subset Q$.

contradiction, if $x \in A$ then since $x \notin B$, we have $x \in A \backslash B = \emptyset$, and this is not possible. So $X \backslash B \subset X \backslash A$.

• (5) $\implies$ (1): Suppose that $X \backslash B \subset X \backslash A$ and prove that $A \subset B$. Let $a \in A$ and prove that $a \in B$. Since $a \in A$, we have $a \in X$. If $a \notin B$, then $a \in X \backslash B \subset X \backslash A$ so that $a \in X \backslash A$ from which we deduce that $a \notin A$, contradiction. So $a \in B$ and $A \subset B$. □

Exercise 4 (Equivalent characterizations of injection)**.** Let E be a set and $f : E \to E$ a function. Prove that the following are equivalent:

(1) f is injective;
(2) $\forall X \subset E, \quad f^{-1}(f(X)) = X$;
(3) $\forall X, Y \subset E, \quad f(X \cap Y) = f(X) \cap f(Y)$;
(4) $\forall X, Y \subset E, \quad X \cap Y = \emptyset \implies f(X) \cap f(Y) = \emptyset$;
(5) $\forall X, Y \subset E, \quad Y \subset X \implies f(X \backslash Y) = f(X) \backslash f(Y)$.

See the previous exercise for definitions of image and preimage.

Careful! The goal of this exercise is to prove five implications as in the example. If you try to prove individually each statement, you will not succeed since taken alone they are not true in general! They are only true when at least one of the statement out of the five is assumed true.

(1) $\implies$ (2): Suppose that f is injective and let $X \subset E$. The direction $X \subset f^{-1}(f(X))$ is always true so we prove that $f^{-1}(f(X)) \subset X$. Let $x \in f^{-1}(f(X))$. Since $y = f(x) \in f(X)$, there exists $z \in X$ such that $y = f(z)$. By injectivity of f, $f(x) = f(z)$ implies that $x = z$. But since $z \in X$, we have that $x \in X$.

(2) $\implies$ (3): Suppose (2) and let $X, Y \subset E$. The direction $f(X \cap Y) \subset f(X) \cap f(Y)$ is always true so we prove that $f(X) \cap f(Y) \subset f(X \cap Y)$. This is not easy question, but one should try doing and not give in easily: the point is that if you apply the definitions correctly and try enough times without making mistakes, your chance of success can only increase during time.

• Let $z \in f(X) \cap f(Y)$, so that there exist $x \in X$ and $y \in Y$ such that $z = f(x)$ and $z = f(y)$. We want to show that $z \in f(X \cap Y)$ so we want to produce an element $u \in X \cap Y$ such that $z = f(u)$. Actually we are going to prove that $x = y$ and take $u = x \in X \cap Y$. Consider the set $A = \{x\}$, and since $f(x) = z$, $f(A) = \{z\}$ and $f^{-1}(f(A)) = A = \{x\}$. Since $y \in f^{-1}(\{z\})$, we have that $y \in f^{-1}(f(A)) = \{x\}$ and we conclude that $y = x$. Finally, taking $u = x = y \in X \cap Y$ shows that $z \in f(X \cap Y)$ since $u \in X \cap Y$. In sum the difficulty is to choose the right set A to which apply the assumption (2).

• Another quicker way (but as you have seen "quick" is very relative) consists of using Exercise 3.(4) to $C = f(X)$ and $D = f(Y)$ to write $f^{-1}(f(X) \cap f(Y)) = f^{-1}(f(X)) \cap f^{-1}(f(Y)) = X \cap Y$ by assumption. Then we apply f on both side and use the fact that we always have $A \subset f(f^{-1}(A))$ for arbitrary set A to conclude that $f(X \cap Y) = f(f^{-1}(f(X) \cap f(Y))) \subset f(X) \cap f(Y)$.

• As proposed by Julian we can also argue by contradiction. Assume that there exists $z \in f(X) \cap f(Y)$ that is not in $f(X \cap Y)$. Since $z \in f(X)$ then there exists $x \in X$ such that $z = f(x)$. Since $z \in f(Y)$ and $f(x) = z$, it follows that $x \in f^{-1}(f(Y)) = Y$ so that $x \in X \cap Y$. Then $z = f(x) \in f(X \cap Y)$. This is a contradiction.

(3) $\implies$ (4): Suppose (3) and let $X, Y \subset E$ such that $X \cap Y = \emptyset$. Since $f(\emptyset) = \emptyset$, $f(X \cap Y) = \emptyset$ and by (3), $f(X) \cap f(Y) = f(X \cap Y) = \emptyset$. The implication follows.

(4) $\implies$ (5): Suppose (4) and let $X, Y \subset E$ such that $Y \subset X$.

• A "quicker" way is to manipulate directly on sets. Notice that $X = Y \cup (X \backslash Y)$ and that $Y \cap (X \backslash Y) = \emptyset$. Applying (4) to Y and $X \backslash Y$ yields $f(Y) \cap f(X \backslash Y) = \emptyset$, and since by Exercise 3.(1), $f(X) = f(Y) \cup f(X \backslash Y)$, we conclude that by substracting $f(Y)$ that $f(X \backslash Y) = f(X) \backslash f(Y)$.

• A safer way is to go back to elements and prove two inclusions. If $Y = X$ then the claim is evident (we do this to make sure that we can pick z in the next phrase).

First, take $z \in f(X \backslash Y)$ and prove that $z \in f(X) \backslash f(Y)$. Since $z \in f(X \backslash Y)$, there exists some $x \in X$, $x \notin Y$ such that $z = f(x)$. It follows that $z \in f(X)$. To see that $z \notin f(Y)$, suppose the contrary and let $y \in Y$ such that $z = f(y)$. Then $Y \cap (X \backslash Y) = \emptyset$ but $f(Y) \cap f(X \backslash Y) \supset \{z\} \neq \emptyset$, since $z \in f(Y)$ (because $z = f(y)$) and $z \in f(X \backslash Y)$ (because $z = f(x)$). This contradiction shows that $z \notin f(Y)$ and hence $z \in f(X) \backslash f(Y)$.

Then take $z \in f(X) \backslash f(Y)$ and show that $z \in f(X \backslash Y)$. Since $z \in f(X)$, there exists $x \in X$ such that $z = f(x)$. Now $x \notin Y$ because otherwise $z \in f(Y)$ which is impossible by the assumption. Hence $x \in X \backslash Y$ and $z = f(x) \in f(X \backslash Y)$.

(5) $\implies$ (1): Suppose (5) and let $x, y \in E$ such that $f(x) = f(y)$. By contradiction suppose that $x \neq y$, and consider $Y = \{y\}$ and $X = \{x, y\}$. We have $f(X) = f(Y) = f(X \backslash Y)$ so by (5), $f(X \backslash Y) = f(X) \backslash f(Y) = \emptyset$. This is impossible, so if $f(x) = f(y)$ then necessarily we have $x = y$: this yields the injectivity of f.

4. Give names to objects

4.1. **Cite the theorems.** To use a theorem, one should check the conditions, clearly cite the theorem then write the correct deduction.

Exercise 5 (The so-called theorem of "adjacent sequences")**.** Let (a_n), (b_n) be two real sequences satisfying the following conditions:

(1) (a_n) is increasing;
(2) (b_n) is decreasing;
(3) $(b_n - a_n)$ converges to 0.

The theorem states that (a_n) and (b_n) converge and to the same limit.

The exercise is about completing the steps in the proof by carefully writing the conditions and citing the right theorems.

Proof. Let (a_n), (b_n) be two adjacent sequences.

• First we show that for all $n > 0$, $a_n \leq b_n$. Since that $(b_n - a_n)$ is decreasing and converges to 0, [. . . (A). . .], $(b_n - a_n)$ is a positive sequence.
• [. . . (B). . .], it follows that (a_n) is bounded from above and (b_n) is bounded from below.
• [. . . (C). . .], (a_n) converges to some $\alpha \in \mathbb{R}$ and (b_n) converges to some $\beta \in \mathbb{R}$.
• Since $(b_n - a_n)$ converges to 0, [. . . (D). . .], $\alpha = \beta$. This finishes the proof of the theorem. ☐

Write down the arguments (A), (B), (C), (D) in the proof. Give an example of a pair of adjacent sequences.

4.2. **Reader-friendly proof.** One way to write reader-friendly proof is to remember to give clear names to any new objects appearing in the proof.

Exercise 6 (Another proof of Bolzano-Weierstrass)**.** Try to understand the argument below and write a clear proof of Bolzano-Weierstrass' theorem by clearly naming all the objects that appear.

Alice claims to have a proof of Bolzano-Weierstrass using the theorem in the previous exercise. She says:

> "I'll take a closed bounded interval and an infinite set of elements inside this interval. First I cut the interval into two equal parts, and since the set is infinite one half must contain an infinite number of elements of the set. Now I take this half and cut it again into two equal parts, and one of them must contain an infinite number of elements of the set. I repeat this cutting many many times, my interval gets smaller and smaller and in the end it will reduce to a point: this will be a point of accumulation. [...] Oh and you ask me why does it reduce to a point in the end? Well just take the end-points of the intervals that I choose each time, they form two adjacent sequences, right?"

Try to write a reader-friendly and rigorous version of Alice's argument!