Dependence Logic

Exercise 12

Note: The exercise class is postponed to Dec 13 due to Independence Day on Dec 6.

In these exercises, the letters $\mathsf{u}, \mathsf{v}, \mathsf{w}, \mathsf{x}, \mathsf{y}, \mathsf{z}, \dots$ in serif font stand for arbitrary (possibly empty) sequences of first-order variables.

Exercise 1. Let $\alpha(\mathsf{x})$ be a first-order formula. Prove that

$$\dot\sim \alpha(\mathsf{x}) \equiv \exists \mathsf{v}(\mathsf{v} \subseteq \mathsf{x} \wedge \neg\alpha(\mathsf{v}/\mathsf{x})).$$

Exercise 2. Prove the following:

(1) $\dot\sim {=}(\mathsf{x}, \mathsf{y}) \equiv \exists \mathsf{u}_0\mathsf{v}_0\mathsf{u}_1\mathsf{v}_1(\mathsf{u}_0\mathsf{v}_0 \subseteq \mathsf{xy} \;\wedge\; \mathsf{u}_1\mathsf{v}_1 \subseteq \mathsf{xy} \;\wedge\; \mathsf{u}_0 = \mathsf{u}_1 \;\wedge\; \mathsf{v}_0 \neq \mathsf{v}_1)$,

(2) $\dot\sim \mathsf{x} \subseteq \mathsf{y} \equiv \exists^1 \mathsf{v}(\mathsf{v} \subseteq \mathsf{x} \;\wedge\; \mathsf{v} \neq \mathsf{y})$,

Exercise 3. Consider the following rule:

$$\dfrac{\begin{array}{ccc}[\mathsf{y} \subseteq \mathsf{x}] & & [\neg\alpha(\mathsf{y})] \\ & D & \\ & \vdots & \\ & \bot & \end{array}}{\alpha(\mathsf{x}/\mathsf{y})} \subseteq\mathsf{Cmp}^{\neg} \quad (1)$$

(1) α is first-order, and y does not occur freely in any undischarged assumptions in D.

(1) Show that the rule $\subseteq\mathsf{Cmp}^{\neg}$ is sound in inclusion logic FO($\subseteq$).

(2) Show that the rule $\dot\sim\mathsf{RAA}$ restricted to first-order formulas is derivable in the system of FO($\subseteq$) extended with the rule $\subseteq\mathsf{Cmp}^{\neg}$, that is, show that $\Gamma, \dot\sim \alpha \vdash \bot$ implies $\Gamma \vdash \alpha$, whenever α is first-order.

Exercise 4. Let $\mathcal{D}$ be a set of upwards closed generalized dependencies, and let $\phi \in \mathrm{FO}(\mathcal{D})$ be a formula that contains no first-order literals. Show that ϕ is upwards closed, i.e., for all structures $\mathcal{M}$ and teams X, Y of $\mathcal{M}$:

$$\mathcal{M} \models_X \phi \text{ and } X \subseteq Y \implies \mathcal{M} \models_Y \phi.$$

Exercise 5. Consider the following upwards closed dependencies.

- Totality: $\mathcal{M} \models \mathtt{All}(\vec{x})$ if and only if $|X[\vec{x}]| = |\mathsf{dom}(\mathcal{M})|^{|\vec{x}|}$;

- Non-dependence: $\mathcal{M} \models_X \neq(\vec{x}, y)$ if and only if there exist $s, s' \in X$ with $s(\vec{x}) = s'(\vec{x})$ but $s(y) = s'(y)$.

Show that both totality and non-dependence are first-order definable.

Exercise 6. Let $\mathcal{D}$ be a set of dependencies and let $\phi(\vec{x})$ be an $\mathrm{FO}(=(.), \mathcal{D})$ formula with free variables from $\vec{x} = (x_1, \ldots, x_n)$. Show that $\phi(\vec{x})$ is equivalent to some formula of the form

$$\exists \vec{y}(=(\vec{y}) \wedge \psi(\vec{x}, \vec{y})),$$

where $\psi \in \mathrm{FO}(\mathcal{D})$, and $\vec{x}$ and $\vec{y}$ do not share any variables.

Exercise 7. Consider the following dependencies.

- Non-exclusion: $\mathcal{M} \models_X \vec{x} \nmid \vec{y}$ if and only if there exist $s, s' \in X$ with $s(\vec{x}) = s'(\vec{y})$;
- Inconstancy: $\mathcal{M} \models_X \neq(x)$ if and only if $|X[\vec{x}]| > 1$;
- Constancy: $\mathcal{M} \models =(x)$ if and only if $|X[\vec{x}]| = 1$.

Let x and y be two variables. Show that $x \nmid y$ is equivalent to some formula $\psi \in \mathrm{FO}(=(.), \neq(.))$. In the proof you may apply any results from the lecture notes. Note that $\mathrm{FO}(=(.), \neq(.))$ is the extension of first-order logic with constancy and inconstancy atoms.