MAST31213 Complexity theory
Master's Programme in Mathematics and Statistics
Fall 2019
Exercise set 13 (for Monday 9.12.)

Read chapters 4.3 – 5.2 and 5.5 of the book.

Exercise 1. Prove Corollary 4.21: For every space constructible $S(n) > \log n$, $\mathbf{NSPACE}(S(n)) = \mathbf{coNSPACE}(S(n))$.

Exercise 2. We defined $\mathbf{\Pi}_i^p = \mathbf{co\Sigma}_i^p$. Show that the following gives an equivalent definition:

For $i \geq 1$, a language L is in $\mathbf{\Pi}_i^p$ if there exists a polynomial-time Turing machine M and a polynomial q such that

$$x \in L \quad \Leftrightarrow \quad \forall u_1 \in \{0,1\}^{q(|x|)} \exists u_2 \in \{0,1\}^{q(|x|)} \cdots Q_i u_i \in \{0,1\}^{q(|x|)} M(x, u_1, \ldots, u_i) = 1$$

where Q_i denotes $\forall$ or $\exists$ depending on whether i is odd or even, respectively.

Exercise 3. Show that the language $\Sigma_i\mathtt{SAT}$ is complete for $\mathbf{\Sigma}_i^p$ under polynomial-time reductions.

Exercise 4. The class **DP** is defined as the set of languages L for which there are two languages $L_1 \in \mathbf{NP}$ and $L_2 \in \mathbf{coNP}$ such that $L = L_1 \cap L_2$. (Note, this is not the same as $\mathbf{NP} \cap \mathbf{coNP}$.)

$\mathtt{EXACT\ INDSET} = \{\llcorner(G,k)\lrcorner : G$ is a graph whose largest independent set has size exactly $k\}$.

Show that

(a) $\mathtt{EXACT\ INDSET} \in \mathbf{\Pi}_2^p$.
(b) $\mathtt{EXACT\ INDSET} \in \mathbf{DP}$.
(c) Every language in **DP** is polynomial-time reducible to `EXACT INDSET`.

Exercise 5. Suppose A is some language such that $\mathbf{P}^A = \mathbf{NP}^A$. Then show that $\mathbf{PH}^A \subseteq \mathbf{P}^A$ (in other words, the proof of Theorem 5.4 *relativizes*.