Dependence Logic I – Selected Solved Exercises

Davide Emilio Quadrellaro

17th December 2019

This file contains detailed solutions to some of the exercises we solved together this semester. If you have any questions, you can contact me at davide.quadrellaro@gmail.com. I whish you success at the exam!

Exercise 3.5

Suppose that $x \notin Fv(\psi)$, we prove that $\exists x(\phi \wedge \psi) \equiv \exists x\phi \wedge \psi$ and $\exists x(\phi \vee \psi) \equiv \exists x\phi \vee \psi$.

Part A: Suppose $x \notin Fv(\psi)$, then we want to show that $\exists x(\phi \wedge \psi) \equiv \exists x\phi \wedge \psi$. First we remark that for any team X and function $F : X \to \wp + (M)$, we have by locality $\mathcal{M} \vDash_X \psi$ iff $\mathcal{M} \vDash_{X(F/x)} \psi$, as $x \notin Fv(\psi)$ and so $X \upharpoonright Fv(\psi) = X(F/x) \upharpoonright Fv(\psi)$. We then get, for any team X, the following equivalences:

$$\begin{aligned}
\mathcal{M} \vDash_X \exists x(\phi \wedge \psi) &\Leftrightarrow \mathcal{M} \vDash_{X(F/x)} \phi \wedge \psi \text{ for some function } F : X \to \wp^+(M) \\
&\Leftrightarrow \mathcal{M} \vDash_{X(F/x)} \phi \text{ and } \mathcal{M} \vDash_{X(F/x)} \psi \text{ for some function } F : X \to \wp^+(M) \\
&\Leftrightarrow \mathcal{M} \vDash_X \exists x\phi \text{ and } \mathcal{M} \vDash_{X(F/x)} \psi, \text{ by the semantics of } \exists \\
&\Leftrightarrow \mathcal{M} \vDash_X \exists x\phi \text{ and } \mathcal{M} \vDash_X \psi, \text{ by locality} \\
&\Leftrightarrow \mathcal{M} \vDash_X \exists x\phi \wedge \psi.
\end{aligned}$$

Which suffices to prove our claim. □

Part B: Suppose $x \notin Fv(\psi)$, then we want to show that $\exists x(\phi \vee \psi) \equiv \exists x\phi \vee \psi$. First we remark that for any team X and function $F : X \to \wp + (M)$, we have by locality $\mathcal{M} \vDash_X \psi$ iff $\mathcal{M} \vDash_{X(F/x)} \psi$, as $x \notin Fv(\psi)$ and so $X \upharpoonright Fv(\psi) = X(F/x) \upharpoonright Fv(\psi)$. We then have the two following directions.

($\Rightarrow$) We first show that $\exists x(\phi \vee \psi) \vDash \exists x\phi \vee \psi$:

$$\begin{aligned}
\mathcal{M} \vDash_X \exists x(\phi \vee \psi) &\Rightarrow \mathcal{M} \vDash_{X(F/x)} \phi \vee \psi, \text{ for some function } F : X \to \wp^+(M) \\
&\Rightarrow \mathcal{M} \vDash_Y \phi \text{ and } \mathcal{M} \vDash_Z \psi, \text{ with } Y \cup Z = X(F/x).
\end{aligned}$$

Now let $Y' = Y \upharpoonright dom(X)$ and $Z' = Z \upharpoonright dom(X)$, then since $Y \cup Z = X(F/x)$ it follows that $Y' \cup Z' = X$. We then define $F' : Y' \to \wp^+(M)$ as $F'(s) = F(s) \cap Y[x]$, hence it is clearly the case that $Y'(F'/x) = Y$. Therefore we get:

$$\mathcal{M} \vDash_{Y'(F'/x)} \phi \text{ and } \mathcal{M} \vDash_Z \psi, \text{ with } Y \cup Z = X$$

By locality and the semantics of existential quantifier we then have that $\mathcal{M} \vDash_{Y'} \exists x\phi$ and $\mathcal{M} \vDash_{Z'} \psi$, with $Y' \cup Z' = X$. Therefore, we finally get that $\mathcal{M} \vDash_X \exists x\phi \vee \psi$.

($\Leftarrow$) We now prove that $\exists x\phi \vee \psi \vDash \exists x(\phi \vee \psi)$:

$$\begin{aligned}\mathcal{M} \vDash_X \exists x\phi \vee \psi &\Rightarrow \mathcal{M} \vDash_Y \exists x\phi \text{ and } \mathcal{M} \vDash_Z \psi, \text{ with } Y \cup Z = X\\ &\Rightarrow \mathcal{M} \vDash_{Y(F/x)} \phi \text{ and } \mathcal{M} \vDash_Z \psi, \text{ with } F : Y \to \wp^+(M).\end{aligned}$$

Now, we define $F' : X \to \wp + (M)$ as $F'(s) = F(s)$, if $s \in Y$, otherwise $F'(s) = a$, where $a \in M$ is an arbitrary element of the underlying model. Then we clearly have that $F \upharpoonright Y = F$ and so that $Y(F'/x) = Y(F/x)$. Moreover, since $x \notin Fv(\psi)$, we then obtain that:

$$\mathcal{M} \vDash_{Y(F'/x)} \phi \text{ and } \mathcal{M} \vDash_{Z(F'/x)} \psi.$$

Finally, since $Y \cup Z = X$ we get that $Y(F'/x) \cup Z(F'/x) = X(F'/x)$ and thus we have $\mathcal{M} \vDash_{X(F'/x)} \phi \vee \psi$, which finally entails $\mathcal{M} \vDash_X \exists x(\phi \vee \psi)$. □

Exercise 9.3

Let ψ be a formula of the form $[LFP_{x,R}\phi(x,y)]x$, where $\phi \in FO$. We show that ψ is equivalent to the formula $[LFP_{xy,R'}\phi'(x,y)]xy$, where $\phi' = \phi[R'(t,y)/R(t)]$. We first prove two important claims.

Claim 1: Let $S \subseteq M$, $S' \subseteq M^2$ such that for all assignement s we have that $s(x) \in S \Leftrightarrow s(xy) \in S'$, then we have that $(\mathcal{M}, S) \vDash_s \phi(x, s(y)) \Leftrightarrow (\mathcal{M}, S') \vDash_s \phi(x, y)$. We show this by induction on ϕ. If ϕ is an atomic formula and it does not contain R then obviously $(\mathcal{M}, S) \vDash_s \phi(x, s(y)) \Leftrightarrow (\mathcal{M}, S') \vDash_s \phi(x, y)$. If ϕ is of the form $R(x)$, then we have $(\mathcal{M}, S) \vDash_s R(x) \Leftrightarrow s(x) \in S \Leftrightarrow s(xy) \in S' \Leftrightarrow (\mathcal{M}, S') \vDash_s \phi(x, y)$. The induction steps follow by routine proof. □

We now also prove the next claim:

Claim 2: We prove by transfinite induction on $\alpha \in Ord$ that:

$$s(x) \in \Gamma^{\alpha}_{\mathcal{M},\phi(x,s(y))} \Leftrightarrow s(xy) \in \Gamma^{\alpha}_{\mathcal{M},\phi'(x,y)}.$$

- Suppose $\alpha = 0$. Then we have:

$$\begin{aligned}s(x) \in \Gamma^0_{\mathcal{M},\phi(x,s(y))} &\Leftrightarrow (\mathcal{M}, \Gamma^0_{\mathcal{M},\phi(x,s(y))}) \vDash_s \phi(x, s(y))\\ &\Leftrightarrow (\mathcal{M}, \emptyset) \vDash_s \phi(x, s(y))\\ &\Leftrightarrow (\mathcal{M}, \emptyset) \vDash_s \phi'(x, y) && \text{(by Claim 1)}\\ &\Leftrightarrow (\mathcal{M}, \Gamma^0_{\mathcal{M},\phi'(x,y)}) \vDash_s \phi'(x, y)\\ &\Leftrightarrow s(xy) \in \Gamma^0_{\mathcal{M},\phi'(x,y)}.\end{aligned}$$

- Suppose $\alpha = \beta + 1$ is a successor ordinal. Notice that by induction hypothesis we have that $s(x) \in \Gamma^{\beta}_{\mathcal{M},\phi(x,s(y))} \Leftrightarrow s(xy) \in \Gamma^{\beta}_{\mathcal{M},\phi'(x,y)}$. Then we have:

$$\begin{aligned}s(x) \in \Gamma^{\alpha}_{\mathcal{M},\phi(x,s(y))} &\Leftrightarrow s(x) \in \Gamma_{\mathcal{M},\phi(x,s(y))}(\Gamma^{\beta}_{\mathcal{M},\phi(x,s(y))})\\ &\Leftrightarrow s(x) \in \{s(x) \in M : (\mathcal{M}, \Gamma^{\beta}_{\mathcal{M},\phi(x,s(y))}) \vDash_s \phi(x, s(y))\}\\ &\Leftrightarrow s(xy) \in \{s(xy) \in M^2 : (\mathcal{M}, \Gamma^{\beta}_{\mathcal{M},\phi'(x,y)}) \vDash_s \phi'(x, y)\} && \text{(by Claim 1 \& I.H.)}\\ &\Leftrightarrow s(xy) \in \Gamma_{\mathcal{M},\phi'(x,y)}(\Gamma^{\beta}_{\mathcal{M},\phi'(x,y)})\\ &\Leftrightarrow s(xy) \in \Gamma^{\alpha}_{\mathcal{M},\phi'(x,y)}.\end{aligned}$$

- If $\alpha = \lambda$ is a limit ordinal, then we have:

$$\begin{aligned} s(x) \in \Gamma^{\lambda}_{\mathcal{M},\phi(x,s(y))} &\Leftrightarrow s(x) \in \Gamma^{\beta}_{\mathcal{M},\phi(x,s(y))} \text{ for some } \beta < \lambda \\ &\Leftrightarrow s(xy) \in \Gamma^{\beta}_{\mathcal{M},\phi'(x,y)} \text{ for some } \beta < \lambda \quad \text{(by induction hypothesis)} \\ &\Leftrightarrow s(xy) \in \Gamma^{\lambda}_{\mathcal{M},\phi'(x,y)}. \qquad \square \end{aligned}$$

Finally, we have that:

$$\begin{aligned} \mathcal{M} \vDash_s [LFP_{x,R}\phi(x,y)]x &\Leftrightarrow s(x) \in \Gamma^{\infty}_{\mathcal{M},\phi(x,s(y))} \\ &\Leftrightarrow s(xy) \in \Gamma^{\infty}_{\mathcal{M},\phi'(x,y))} \qquad \text{(by Claim 2)} \\ &\Leftrightarrow \mathcal{M} \vDash_s [LFP_{x,R'}\phi(x,y)]xy. \end{aligned}$$

Exercise 9.4

We complete the induction proof of Theorem 6.3.6 for universal quantification by showing that for all $\mathcal{M}$ and X, $\mathcal{M} \vDash_X \psi^+(\vec{x},\vec{y})$ iff for all $s \in X$, $(\mathcal{M}, X[\vec{x}]) \vDash_s \psi(R,\vec{x},\vec{y})$, where $\psi(R,\vec{x},\vec{y}) = \forall v\alpha(R,\vec{x},\vec{y},v)$, and $\psi^+(\vec{x},\vec{y}) = \forall v\alpha^+(\vec{x},\vec{y},v)$. The induction hypothesis then tells us that for all $\mathcal{M}$ and Y, $\mathcal{M} \vDash_Y \alpha^+(\vec{x},\vec{y},v)$ iff for all $s \in Y$, $(\mathcal{M}, Y[\vec{x}]) \vDash_s \alpha(R,\vec{x},\vec{y},v)$.

We only prove the left to right direction, as the right to left direction follows analogously. Suppose that $\mathcal{M} \vDash_X \psi^+(\vec{x},\vec{y})$, then we have that $\mathcal{M} \vDash_X \forall v\alpha^+(\vec{x},\vec{y},v)$, hence $\mathcal{M} \vDash_{X(M/v)} \alpha^+(\vec{x},\vec{y},v)$. Then, it follows by our induction hypothesis that we get, for all $s \in X(M/v)$, that $(\mathcal{M}, X(M/v)[\vec{x}]) \vDash_s \alpha(R,\vec{x},\vec{y},v)$. Now, it is clear that $X(M/v)[\vec{x}] = X[\vec{x}]$, thus we have that $(\mathcal{M}, X[\vec{x}]) \vDash_s \alpha(R,\vec{x},\vec{y},v)$ holds for all $s \in X(M/v)$. Finally, this means that for all $s \in X$ and $a \in M$, we have $(\mathcal{M}, X[\vec{x}]) \vDash_{s(a/v)} \alpha(R,\vec{x},\vec{y},v)$, which means $(\mathcal{M}, X[\vec{x}]) \vDash_{s(a/v)} \forall v\alpha(R,\vec{x},\vec{y},v)$, thus $(\mathcal{M}, X[\vec{x}]) \vDash_s \forall v\psi(R,\vec{x},\vec{y})$. $\square$

Exercise 10.5

Part A: We prove the following lemma: for any formula ϕ and any term t which is free for x, we have that for any team X of a model $\mathcal{M}$, $\mathcal{M} \vDash_X \phi(t/x) \Leftrightarrow \mathcal{M} \vDash_{X(t/x)} \phi$, where $X(t/x) = \{s(s(t^{\mathcal{M}})/x) : s \in X\}$. We prove this claim by induction on the complexity of ϕ:

- If ϕ is an atomic first order formulas of the form $x = y$, then we have:

$$\begin{aligned} \mathcal{M} \vDash_X x = y(t/x) &\Leftrightarrow \mathcal{M} \vDash_X t = y \\ &\Leftrightarrow \forall s \in X, s(t) = s(y) \\ &\Leftrightarrow \forall s \in X(t/x), s(x) = s(t) = s(y) \\ &\Leftrightarrow \mathcal{M} \vDash_{X(t/x)} x = y. \end{aligned}$$

- If ϕ is an atomic first order formulas of the form $x = y$, then we proceed analogously.

- If ϕ is an atomic first order formulas of the form $R(x\vec{y})$, then we have:

$$\begin{aligned} \mathcal{M} \vDash_X R(x\vec{y})(t/x) &\Leftrightarrow \mathcal{M} \vDash_X R(t\vec{y}) \\ &\Leftrightarrow \forall s \in X, s(t\vec{y}) \in R^{\mathcal{M}} \\ &\Leftrightarrow \forall s \in X(t/x), s(x\vec{y}) = s(t\vec{y}) \in R^{\mathcal{M}} \\ &\Leftrightarrow \mathcal{M} \vDash_{X(t/x)} R(x\vec{y}). \end{aligned}$$

- If ϕ is an atomic first order formulas of the form $\neg R(x\vec{y})$, then we proceed analogously.
- If ϕ is an dependence atom of the form $=\!(\vec{y}, x)$, then we have:

$$\begin{aligned}\mathcal{M} \vDash_X =\!(\vec{y}, x)(t/x) &\Leftrightarrow \mathcal{M} \vDash_X =\!(\vec{y}, t)\\ &\Leftrightarrow \forall s, s' \in X, \text{ if } s(\vec{y}) = s'(\vec{y}) \text{ then } s(t) = s'(t)\\ &\Leftrightarrow \forall s, s' \in X(t/x), \text{ if } s(\vec{y}) = s'(\vec{y}) \text{ then } s(x) = s(t) = s'(t) = s(x)\\ &\Leftrightarrow \mathcal{M} \vDash_{X(t/x)} =\!(\vec{y}, x).\end{aligned}$$

  If ϕ is of the form $=\!(x, \vec{y})$, or other similar configurations, then we proceed analogously.
- If ϕ is an inclusion, exclusion or independence atom, then we proceed analogously.
- If ϕ is of the form $\psi \wedge \chi$, then we have:

$$\begin{aligned}\mathcal{M} \vDash_X (\psi \wedge \chi)(t/x) &\Leftrightarrow \mathcal{M} \vDash_X \psi(t/x) \text{ and } \mathcal{M} \vDash_X \chi(t/x)\\ &\Leftrightarrow \mathcal{M} \vDash_{X(t/x)} \psi \text{ and } \mathcal{M} \vDash_{X(t/x)} \chi, \text{ by I.H.}\\ &\Leftrightarrow \mathcal{M} \vDash_{X(t/x)} \psi \wedge \chi.\end{aligned}$$

- If ϕ is of the form $\psi \vee \chi$, then we have:

$$\begin{aligned}\mathcal{M} \vDash_X (\psi \vee \chi)(t/x) &\Leftrightarrow \mathcal{M} \vDash_Y \psi(t/x) \text{ and } \mathcal{M} \vDash_Z \chi(t/x), \text{ with } Y, Z \subseteq X \text{ s.t. } Y \cup Z = X\\ &\Leftrightarrow \mathcal{M} \vDash_{Y(t/x)} \psi \text{ and } \mathcal{M} \vDash_{Z(t/x)} \chi, \text{ by I.H.}\\ &\Leftrightarrow \mathcal{M} \vDash_{X(t/x)} \psi \vee \chi, \text{ since } Y \cup Z = X \Leftrightarrow Y(t/x) \cup Z(t/x) = X(t/x).\end{aligned}$$

- If ϕ is of the form $\exists y \psi$, then we have:

$$\begin{aligned}\mathcal{M} \vDash_X (\exists y \psi)(t/x) &\Rightarrow \mathcal{M} \vDash_X \exists y(\psi(t/x)) \text{ since } t \text{ is free for } x\\ &\Rightarrow \mathcal{M} \vDash_{X(F/y)} \psi(t/x), \text{ for some function } F : X \to \wp^+(\mathcal{M})\\ &\Rightarrow \mathcal{M} \vDash_{X(F/y)(t/x)} \psi, \text{ by I.H.}\end{aligned}$$

  Now define $F' : X(t/x) \to \wp^+(\mathcal{M})$ as $F'(s(t/x)) = F(s)$, then we clearly have that $X(F/y)(t/x) = X(t/x)(F'/y)$ and thus:

$$\begin{aligned}&\Rightarrow \mathcal{M} \vDash_{X(t/x)(F'/y)} \psi\\ &\Rightarrow \mathcal{M} \vDash_{X(t/x)} \exists y \psi.\end{aligned}$$

  The other direction follows analogously, by defining, given $F' : X(t/x) \to \wp^+(\mathcal{M})$, the function $F : X \to \wp^+(\mathcal{M})$, such that $F'(s(t/x)) = F(s)$.
- If ϕ is of the form $\forall y \psi$, then we have:

$$\begin{aligned}\mathcal{M} \vDash_X (\forall y \psi)(t/x) &\Leftrightarrow \mathcal{M} \vDash_X \forall y(\psi(t/x)) \text{ since } t \text{ is free for } x\\ &\Leftrightarrow \mathcal{M} \vDash_{X(M/y)(t/x)} \psi, \text{ by I.H.}\\ &\Leftrightarrow \mathcal{M} \vDash_{X(t/x)(M/y)} \psi, \text{ since } X(M/y)(t/x) = X(t/x)(M/y)\\ &\Leftrightarrow \mathcal{M} \vDash_{X(t/x)} \forall y \psi. \ \square\end{aligned}$$

Part B: We use the result just shown to prove that the rule $\forall$E of dependence logic is sound. It suffices to show that $\forall x \phi \vDash \phi(t/x)$. Suppose that $\mathcal{M} \vDash_X \forall x \phi$, then it follows that $\mathcal{M} \vDash_{X(M/x)} \phi$. Since we are working in dependence logic, we can use the fact that our logic is downward closed, hence since $X(t/x) \subseteq X(M/x)$, it follows that $\mathcal{M} \vDash_{X(t/x)} \phi$, so by part A of this exercise we get that $\mathcal{M} \vDash_X \phi(t/x)$, which finally proves our claim. $\square$

Exercise 11.2

We present the following derivation in the system of natural deduction.

(i) We derive $\vdash z \subseteq z$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{z=z}\,{=}\mathrm{I}}{\forall x(z=z)}\,\forall\mathrm{I}}{\exists x\forall y(zy\subseteq zx\wedge z=z)}\,\forall{\subseteq}\mathrm{Sim} \qquad \dfrac{[\forall y(zy\subseteq zx\wedge z=z)] \qquad \dfrac{\dfrac{[zy\subseteq zx\wedge z=z]}{zy\subseteq zx}\,\wedge\mathrm{E}}{z\subseteq z}\,{\subseteq}\mathrm{Ctr}}{\forall y(z\subseteq z)}\,\forall\mathrm{Sub}}{\forall y(z\subseteq z)}\,\exists\mathrm{E}}{z\subseteq z}\,\forall\mathrm{E}_0$$

(ii) Given $|x| = |y| = |z|$, we derive $xy \subseteq zz \vdash x = y$:

$$\dfrac{\dfrac{}{z=z}\,{=}\mathrm{I} \qquad xy\subseteq zz}{x=y}\,{\subseteq}\mathrm{Cmp}$$

(iii) We derive $xy \subseteq uv \vdash xyy \subseteq uvv$:

$$\dfrac{\dfrac{xy\subseteq uv}{\exists w(xyw\subseteq uvv)}\,\mathrm{W}_\exists \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[xyw\subseteq uvv]}{yxw\subseteq vuv}\,{\subseteq}\mathrm{Exc}}{ywx\subseteq vvu}\,{\subseteq}\mathrm{Exc}}{yw\subseteq vv}\,{\subseteq}\mathrm{Ctr}}{y=w}\,\text{(ii)} \qquad [xyw\subseteq uvv]}{xyy\subseteq uvv}\,{=}\mathrm{Sub}}{xyy\subseteq uvv}\,\exists\mathrm{E}$$

(iv) We derive $\forall x(y \subseteq x)$:

$$\dfrac{\dfrac{\dfrac{}{x\subseteq x}\,\text{(i)}}{\forall w(xy\subseteq xw)}\,\mathrm{W}_\forall \qquad \dfrac{\dfrac{[xy\subseteq xx]}{yx\subseteq xx}\,{\subseteq}\mathrm{Exc}}{y\subseteq x}\,{\subseteq}\mathrm{Ctr}}{\forall x(y\subseteq x)}\,\forall\mathrm{Sub}$$

Exercise 11.5

We define a new type of atom $x\Upsilon y$ as follows:

$$x\Upsilon y \Leftrightarrow \forall s \in X, \exists s' \in X \text{ s.t. } s(x) = s'(x) \text{ and } s(y) \neq s'(y).$$

These atoms are known as *afunctional dependencies* in database theory.

Part A: First we show that $x\Upsilon y \equiv \exists v(xv \subseteq xy \wedge v \neq y)$. We show the left to right direction only as the right to left direction proceed analogously. Suppose $\mathcal{M} \vDash_X x\Upsilon y$, then by definition we have that $\forall s \in X, \exists s' \in X$ s.t. $s(x) = s'(x)$ and $s(y) \neq s'(y)$. We now need to show that $\mathcal{M} \vDash_X \exists v(xv \subseteq xy \wedge v \neq y)$. We define the function $f : X \to \wp^+(M)$ such that $F(s) = s'(y)$ where s' is such that $s(x) = s'(x)$ and $s(y) \neq s'(y)$. Hence we need to verify that $\mathcal{M} \vDash_{X[M/v]} xv \subseteq xy \wedge v \neq y)$. First, we have that for all $s_0 \in X[M/v]$, $s_0 = s(s'(y)/v)$ for some $s \in X$, hence $s_0(xv) = s(x)s'(y) = s'(x)s'(y)$, by our previous assumption, therefore this establishes $\mathcal{M} \vDash_{X[M/v]} xv \subseteq xy$. Moreover, since for all $s_0 \in X[M/v]$ we have that $s_0 = s(s'(y)/v)$ for some $s \in X$, thus $s_0(y) = s'(y) \neq s(y)$, by our former assumption. Finally, this establishes $\mathcal{M} \vDash_{X[M/v]} v \neq y$ our claim and thus proves our claim.

Part B: we derive the following two rules in the system of natural deduction of inclusion logic. We only write the main steps and we leave to the reader to write the full derivation.

(i) We derive $xy\Upsilon z \vdash x\Upsilon zu$.

$$\begin{aligned} xy\Upsilon z &:= \exists v(xyv \subseteq xyz \land v \neq z) && \\ &\vdash \exists v(xv \subseteq xz \land v \neq z) && (\subseteq \mathsf{Ctr}) \\ &\vdash \exists vw(xvw \subseteq xzu \land v \neq z) && (\subseteq \mathsf{W}_\exists) \\ &\vdash \exists vw(xvw \subseteq xzu \land vw \neq zu) && (\lor \mathsf{I}) \\ &:= x\Upsilon zu. && \end{aligned}$$

(ii) We derive $xy\Upsilon zy \vdash xy\Upsilon z$.

$$\begin{aligned} xy\Upsilon zy &:= \exists uv(xyuv \subseteq xyzy \land uv \neq zy) && \\ &\vdash \exists uv(xyuv \subseteq xyzy \land uv \neq zy \land y = v) && \text{Ex. 2.11.ii above} \\ &\vdash \exists uv(xyuv \subseteq xyzy \land u \neq z) && (\lor \mathsf{E}) \\ &\vdash \exists u(xyu \subseteq xyz \land u \neq z) && (\subseteq \mathsf{Ctr}) \\ &:= xy\Upsilon z. && \end{aligned}$$