WEEK VII: ELEMENTARY FUNCTIONS

Notation: studying the variation of a function means finding all its local extrema and the monotonicity of the function between any two consecutive local extrema.

Problem I. Logarithm function

We study two functions f and g.

A. Study of the function f. Consider the function f defined by $f(0) = 0$ and

$$f(t) = \frac{t}{\ln(t)}.$$

1. Determine the domain of definition of f.
2. Study the derivability of f at 0 using the definition.
3. Calculate f' on $(0, 1)$ then the limit of f' at 0^+.
4. Study the variation of f on its domain of definition.

1. The domain of definition of f is $\mathbb{R}_{>0}\backslash\{1\}$.
2. Using the definition, we should look at

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{\frac{x}{\ln(x)}}{x} = \lim_{x\to 0} \frac{1}{\ln(x)}.$$

Since $\ln(x)$ goes to $-\infty$ as x goes to 0, this limit exists and is equal is 0.

3. For $x \in (0, 1)$, we can calculate the derivative of f and it is equal to

$$f'(x) = \frac{\ln(x) - x \cdot \frac{1}{x}}{\ln^2(x)} = \frac{\ln(x) - 1}{\ln^2(x)} = \frac{1}{\ln(x)} - \frac{1}{\ln^2(x)}.$$

Both quantities goes to 0 as x goes to 0^+ so that the following limit holds:

$$\lim_{x\to 0^+} f'(x) = 0.$$

4. To study the variation of f we look at the sign of f' on its domain of definition. Write

$$f'(x) = \frac{1}{\ln(x)}\left(1 - \frac{1}{\ln(x)}\right).$$

Now look at each term

- $\frac{1}{\ln(x)}$ is negative for $0 < x < 1$ and positive for $1 < x < \infty$;
- $1 - \frac{1}{\ln(x)}$ is positive for $0 < x < 1$, negative for $1 < x < e$, (zero at $x = e$) and positive for $e < x < \infty$.

Finally, by looking at the sign of the product we can conclude that

(1) f is decreasing on the interval $(0, 1)$;
(2) f is decreasing on the interval $(1, e]$;
(3) f is increasing on the interval $[e, \infty)$.

Attention that f is not decreasing on the interval $(0, e)$!

(I've written everything with the variable x instead of t, but of course this is just a personal choice.)

B. Study of the function g. Consider the function g defined on $t > 0$ by

$$g(t) = \frac{t^2 - 1}{t \ln(t)}.$$

1. Determine the function h defined for $t > 0$ by
$$g'(t) = \frac{1 + t^2}{t^2 (\ln)^2(t)} \cdot h(t).$$
2. Study the sign on $\mathbb{R}_{>0}$ of the function
$$t \mapsto \ln(t) + \frac{1 - t^2}{1 + t^2}.$$
3. Study the variation of the function g.
4. Determine the limit of g at 1. You can first write $t = 1 + \epsilon$.
5. Compare the function f with the function g.

1. A calculation yields
$$h(t) = \ln(t) + \frac{1 - t^2}{1 + t^2}.$$

2. This is not an easy question. We can first notice that $h(1) = 0$ then try to study independently the variation of h of the interval $(0, 1]$ and the interval $[1, \infty)$. Anyways it is healthy to have in hand

$$h'(t) = \frac{1}{t} + \frac{-4t}{(1 + t^2)^2}.$$

Actually we can write

$$h'(t) = \frac{(1 + t^2)^2 - 4t^2}{t \cdot (1 + t^2)^2} = \frac{(1 - t^2)^2}{t \cdot (1 + t^2)^2}$$

so that $h(t) \geq 0$ for $t \geq 0$. It follows (from the mean value theorem) that h is negative on $(0, 1]$ and positive on $[1, \infty)$.

3. Since g' has the same sign as h, we know that g is decreasing on $(0, 1]$ and increasing on $[1, \infty)$.

4. Writing $t = 1 + \epsilon$, we have that

$$g(1 + \epsilon) = \frac{2\epsilon + \epsilon^2}{(1 + \epsilon) \cdot \ln(1 + \epsilon)}.$$

Using the limit $\lim\limits_{\epsilon \to 0} \frac{\ln(1+\epsilon)}{\epsilon} = 1$, we have

$$\lim_{\epsilon \to 0} g(1 + \epsilon) = \lim_{\epsilon \to 0} \frac{2\epsilon + \epsilon^2}{(1 + \epsilon) \cdot \epsilon} = 2.$$

5. We can form the difference $(g - f)(t)$:

$$(g - f)(t) = \left(\frac{t^2 - 1}{t} - t \right) \frac{1}{\ln(t)} = -\frac{1}{t \ln(t)}.$$

Studying the sign of $-\frac{1}{t \ln(t)}$, it follows that $f \leq g$ on $(0, 1)$ and $f \geq g$ on $(1, \infty)$.

Problem II. Trigonometric functions

We study some properties related to the function arctan.

A. Study of a function g. Let $g : \mathbb{R} \to \mathbb{R}$ defined as

$$g(x) = \arctan(x) - x + \frac{x^3}{3}.$$

1. Verify that g is an odd function.
2. Calculate the derivative g' on $\mathbb{R}$.
3. Show that for all $x \in \mathbb{R}$,

$$0 \leq g'(x) \leq x^2.$$

4. Deduce that for all $x \in \mathbb{R}_{>0}$

$$x - \frac{x^3}{3} \leq \arctan(x) \leq x.$$

1. Each term in the definition of g is odd.
2. Remembering that $\arctan'(x) = \frac{1}{1+x^2}$ we have that

$$g'(x) = \frac{1}{1+x^2} - 1 + x^2.$$

3. Showing that $g'(x) \leq x^2$ is equivalent to showing that $\frac{1}{1+x^2} \geq 1$: this is easy.

Showing that $g'(x) \geq 0$ is equivalenet to showing that $\frac{1}{1+x^2} \leq 1 - x^2$. This is the same as $1 - x^4 \leq 1$ which is also true.

Other methods include studying the variation of g'.

4. Since $g(0) = 0$ and $g'(x) \geq 0$ for $x \geq 0$, we conclude that $g(x) \geq 0$ for $x > 0$ (by the mean value theorem). This writes

$$\forall x > 0, x - \frac{x^3}{3} \leq \arctan(x).$$

To prove the other inequality, notice that the function $h : x \mapsto x - \arctan(x)$ satisfies $h(0) = 0$ and $h'(x) \geq 0$ for $x \geq 0$. The mean value theorem yields $h(x) \geq 0$ for $x \geq 0$, which is the other required inequality.

B. Study of a function f. Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(0) = 1$ and otherwise for $x \neq 0$,

$$f(x) = \frac{\arctan(x)}{x}.$$

1. Prove that f is even.
2. Use one question above to study the limit

$$\lim_{x \to 0^+} \frac{\arctan(x) - x}{x^2}.$$

3. Deduce that f is derivable at 0 and give the value of $f'(0)$.
4. Calculate f' on $\mathbb{R}_{>0}$.

1. f is even as the quotient of two *odd* functions.
2. Remembering that for all $x \geq 0$ we have

$$x - \frac{x^3}{3} \leq \arctan(x) \geq x, \quad \text{i.e.} \quad 0 \geq \arctan x - x \geq -\frac{x^3}{3}$$

so that for all $x \geq 0$,

$$-\frac{x}{3} \leq \frac{\arctan(x) - x}{x^2} \leq 0.$$

By the squeeze theorem,

$$\lim_{x \to 0^+} \frac{\arctan(x) - x}{x^2} = 0$$

3. By definition we should study

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{\frac{\arctan(x)}{x} - 1}{x} = \lim_{x \to 0^+} \frac{\arctan(x) - x}{x^2} = 0$$

by the question above, so that $f'(0)$ exists and is equal to 0.

4. It is a calculation and the result is (I believe)

$$\frac{\frac{1}{1+x^2}x - \arctan(x)}{x^2}.$$

Problem III. Polynomial functions

For all $n \geq 1$ we consider the equation for $x \in \mathbb{R}_{\geq 0}$:

$$(E_n) \qquad h_n(x) = x^n + x - 1 = 0.$$

A. Study of solutions of E_n. We first study the function $h_n(x)$.

1. Study the solution of E_n for $n = 1$ and $n = 2$.
2. Calculate $h_n(0)$ and $h_n(1)$ for a general n.
3. Determine the derivative of h_n on $\mathbb{R}_{\geq 0}$.
4. Study the variation of h_n on $\mathbb{R}_{\geq 0}$.
5. Show that E_n has a unique solution x_n in $\mathbb{R}_{\geq 0}$ and $x_n \in (0, 1)$ for $n \geq 1$.

1. For $n = 1$ the equation is

$$h_1(x) = 2x - 1 = 0$$

so that the only solution is $x = \frac{1}{2}$.

For $n = 2$ the equation is a second degree polynomial equation

$$h_2(x) = x^2 + x - 1 = 0.$$

Solving this *and choosing the positive solution*, $x = \frac{-1+\sqrt{5}}{2}$.

2. For a general n,

$$h_n(0) = -1, \quad h_n(1) = 1 + 1 - 1 = 1.$$

3. We have that $h_n'(x) = n \cdot x^{n-1} + 1$ on $\mathbb{R}_{\geq 0}$.
4. Since $h_n'(x) > 0$ for all $x \geq 0$, h_n is strictly increaing on $\mathbb{R}_{\geq 0}$.
5. By a question before, $h_n(0)$ and $h_n(1)$ have different signs so that the continuity of h_n yields a solution x_n in $(0, 1)$ by Bolzano's theorem. The strictly monotonicity of h_n garantees the uniqueness of this solution.

B. Study of an auxiliary function. We now study the function

$$f(x) = \frac{\ln(1-x)}{\ln(x)}.$$

1. Determine the domain of definition of f.
2. Determine the limit of f at every extremities of its domain of definition.
3. Calculate $f'(x)$ on its domain of definition and study the variation of f.

1. The domain of definition of f is the interval $I = (0, 1)$.
2. When x goes to 0^+, $f(x)$ goes to 0 and when x goes to 1^-, $f(x)$ goes to $+\infty$.
3. For $x \in (0, 1)$, by a calculation

$$f'(x) = \frac{-\ln(x) \cdot \frac{1}{1-x} - \ln(1-x) \cdot \frac{1}{x}}{\ln^2(x)}.$$

The quantity in the nominator is always strictly positive so that $f'(x) > 0$ for all $x \in (0, 1)$. Consequently f is strictly increasing on the interval $I = (0, 1)$.

C. Study of a sequence. We study the sequence $\{x_n\}$.

1. Prove that $f(x_n) = n$ for all $n \geq 1$.
2. Prove that $\{x_n\}$ is a strictly increasing sequence.

3. Prove that the sequence $\{x_n\}$ converges to a real number L.
4. Determine L.

1. Since x_n is a solution to E_n we have that $x_n^n = 1 - x_n$, so that

$$f(x_n) = \frac{\ln(1 - x_n)}{\ln(x_n)} = \frac{\ln(x_n^n)}{\ln(x_n)} = \frac{n \ln(x_n)}{\ln(x_n)} = n.$$

2. We have that $f(x_n) > f(x_m)$ if $n > m$. Since f is strictly increasing on $(0, 1)$, this shows that $x_n > x_m$ if $n > m$, hence $\{x_n\}$ is a strictly increasing sequence.

3. The sequence $\{x_n\}$ is strictly increasing and bounded by above by the constant 1, by the monotone convergence theorem, x_n converges to a real number L.

4. This is a non-trivial question. We know that by the above question that $L \leq 1$ and we prove that $L = 1$. Indeed,

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} n = +\infty$$

and that if $L < 1$ then by the continuity of f,

$$\lim_{n\to\infty} f(x_n) = f(L) < \infty$$

which is impossible.