Problem set 1, Advanced risk theory, 24.1.2020

Exercise 1.1. Consider functions $f\colon \mathbb{R} \to (0,\infty)$ and $g\colon \mathbb{R} \to (0,\infty)$. The functions are said to be *asymptotically equivalent* (at infinity) if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1. \tag{1}$$

Relation (1) is often abbreviated using notation $f(x) \sim g(x)$, as $x \to \infty$.

a) Suppose $g(x) = e^{-Rx}$ for a fixed $R > 0$ and all $x \in \mathbb{R}$. Show that if $f(x) \sim g(x)$, as $x \to \infty$, then $-\log f(x) \sim Rx$, as $x \to \infty$.

b) Let f and g be as in Part a). Give an example of f such that $-\log f(x) \sim Rx$ holds, but $f(x) \sim g(x)$ does not hold, as $x \to \infty$.

Exercise 1.2. The *exponential index* $\mathcal{E}(\xi)$ of random variable ξ is defined in risk theory as

$$\mathcal{E}(\xi) = \sup\left\{s \ge 0 : \mathbb{E}\left(e^{s\xi}\right) < \infty\right\} = \liminf_{z\to\infty} \frac{-\log \mathbb{P}(\xi > z)}{z}.$$

Let $a \in \mathbb{R}$. Let c_ξ be the cumulant generating function of ξ so that $\mathcal{E}(\xi) = \sup\{s \ge 0 : c_\xi(s) < \infty\}$.

Give an example of a distribution of ξ for which $\mathcal{E}(\xi) \in (0,\infty)$ and $c_\xi(\mathcal{E}(\xi)) = a$. Hint: You can make a slight modification to a distribution with density $f(x) = Ce^{-x}/(1+x^3)$ for $x > 0$ to match the value of $c_\xi(\mathcal{E}(\xi))$ with a.

Exercise 1.3. The *moment index* $\mathbb{I}(\xi)$ of random variable ξ is defined in risk theory as

$$\mathbb{I}(\xi) = \sup\left\{s \ge 0 : \mathbb{E}\left((\xi^+)^s\right) < \infty\right\} = \liminf_{z\to\infty} \frac{-\log \mathbb{P}(\xi > z)}{\log z},$$

where $\xi^+ = \max(0,\xi)$ is the positive part of ξ. Let ξ_1 and ξ_2 be random variables such that

$$\mathbb{I}(\xi_1) = \mathbb{I}(\xi_2) = \alpha, \tag{2}$$

where $\alpha \in (0,\infty)$. Assume further that ξ_1 and ξ_2 are non-negative and independent.

Show that

$$\mathbb{I}(\xi_1\xi_2) = \alpha.$$

Exercise 1.4 (Exercise 1.3 continues)**.** Let ξ_1 and ξ_2 be real-valued random variables that satisfy (2) and $\mathbb{P}(\xi_1 > z) = \mathbb{P}(\xi_2 > z)$ for all large enough z, but which are not necessarily independent.

Show that

$$\mathbb{I}(\max(\xi_1,\xi_2)) = \alpha.$$

Exercise 1.5 (Exercise 1.4 continues)**.** Let ξ_1 and ξ_2 be non-negative random variables that satisfy (2) and $\mathbb{P}(\xi_1 > z) = \mathbb{P}(\xi_2 > z)$ for all large enough z, but which are not necessarily independent.

Show that

$$\mathbb{I}(\xi_1 + \xi_2) = \alpha.$$

Does the result hold if ξ_1 and ξ_2 are allowed to obtain negative values?