WEEK VI: REVISION

1. Study of functions[27pt]

Part I[9pt]. In this part of the exercise we consider the function

$$g(x) = \ln(1+x) - x.$$

1[2pt]. Justify that g is continuous and derivable on the interval $(-1, \infty)$.

Solution. g is obtained by operations on smooth elementary functions: in particular on the interval $(-1, \infty)$, $\ln(1+x)$ is well-defined and smooth. It follows that g is continuous and derivable on the interval $(-1, \infty)$. □

2[3pt]. Calculate the first and second derivatives of g on $(-1, \infty)$.

Solution. Let $x \in (-1, \infty)$. A calculation yields

$$g'(x) = \frac{1}{1+x} - 1.$$

Deriving once more we get

$$g''(x) = -\frac{1}{(1+x)^2}.$$

□

3[3pt]. Find the maximum of g on $(-1, \infty)$ if it exists (with justification).

Solution. Using Rolle's theorem, we should first search for points $x \in (-1, \infty)$ such that $g'(x) = 0$. Solving this yields $x = 0$.

Next we check if $g(0)$ is the maximum of g on $(-1, \infty)$. For example, we observe that for $x \in (-1, 0)$, $g'(x) > 0$ and for $x \in (0, \infty)$, $g'(x) < 0$. A study of thhe monotonicity of g yields that $g(0)$ is indeed the maximum of g on $(-1, \infty)$.

Other ways of seeing this include using the inequality $\ln(1+x) \leq x$, true for all $x \in (-1, \infty)$ and observing that $g(0) = 0$. □

4[1pt]. Verify that (you can use l'Hôpital's rule)

$$\lim_{x \to 0} \frac{g(x)}{x^2} = -\frac{1}{2}.$$

Solution. Applying l'Hôpital's rule (we verify everytime that we have an indeterminate form)

$$\lim_{x \to 0} \frac{g(x)}{x^2} = \lim_{x \to 0} \frac{g'(x)}{2x} = \lim_{x \to 0} \frac{g''(x)}{2} = -\frac{1}{2}.$$

□

Part II$^{[10pt]}$**.** In this part of the exercise we consider the function

$$\overline{f}(x) = \frac{\ln(1+x)}{x}.$$

$1^{[2pt]}$. Find the domain of definition $\mathcal{D}$ of $\overline{f}$. Justify that $\overline{f}$ is derivable on $\mathcal{D}$.

Solution. We need $1 + x > 0$ and $x \neq 0$. Solving this yields $\mathcal{D} = (-1, 0) \cup (0, \infty)$. Then $\overline{f}$ is derivable on $\mathcal{D}$ as composition of elementary functions. □

$2^{[2pt]}$. By examining the derivative $(\ln)'(1)$, determine the value of $a = \lim_{x\to 0} \overline{f}(x)$.

Solution. In short,

$$a = \lim_{x\to 0} \overline{f}(x) = \lim_{x\to 0} \frac{\ln(1+x)}{x} = (\ln)'(1) = \frac{1}{1} = 1.$$

□

- We consider the function $f : \mathcal{D} \cup \{0\} \to \mathbb{R}$ defined as $f(x) = \overline{f}(x)$ for $x \in \mathcal{D}$ and $f(0) = a$.

$3^{[1pt]}$. Justify that f is continuous on the interval $I = (-1, \infty)$.

Solution. We already know that f is continuous on $\mathcal{D}$: we only (and have) to justify the continuity at 0. This is because by a question above, $\lim_{x\to 0} f(x) = \lim_{x\to 0} \overline{f}(x) = a = f(0)$, so that f is also continuous at 0. □

$4^{[2pt]}$. Calculate $f'(x)$ for $x \in \mathcal{D}$.

Solution. A calculation yields (to check) $f'(x) = -\frac{\ln(1+x)}{x^2} + \frac{1}{x(x+1)}$. □

$5^{[1pt]}$. Wrtie down the expression of the derivative $f'(0)$ using the definition.

Solution. It writes

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{\frac{\ln(1+x)}{x} - 1}{x} = \lim_{x\to 0} \frac{\ln(1+x) - x}{x^2}.$$

□

$6^{[2pt]}$. Using the function g, justify that $f'(0)$ exists and calculate its value $b = f'(0)$.

Solution. We recognize Question 4 of Part I and using that we have $b = -\frac{1}{2}$. □

$7^{[\star]}$. Justify that f is a function of class $\mathcal{C}^1$ on the interval I. You can study by l'Hôpital's rule

$$\lim_{x\to 0} \frac{x - (1+x)\ln(1+x)}{x^2(1+x)}.$$

Solution. We should verify that $f'(x)$ in Question 4 converges to the quantity $f'(0)$ in Question 5. This leads to the following verification (by l'Hôpital)

$$\lim_{x\to 0} \frac{x - (1+x)\ln(1+x)}{x^2(1+x)} = \lim_{x\to 0} \frac{1 - 1 - \ln(1+x)}{2x(1+x) + x^2} = \lim_{x\to 0} \frac{\frac{1}{1+x}}{2+6x} = \frac{1}{2}.$$

□

Part III[8pt]**.** We admit that

$$\lim_{x \to \infty} \frac{\ln(x)}{x} = 0.$$

1[2pt]. Let $\alpha \in (0, 1)$ and study the limits

$$A = \lim_{x \to 0} \frac{\ln(1 + x)}{\alpha x}; \quad B = \lim_{x \to \infty} \frac{\ln(1 + x)}{\alpha x}.$$

Solution. By the study of the function f above we know that $A = \frac{1}{\alpha}$. By a change of variables and the admitted limit in the beginning of this part, $B = 0$. $\square$

2[2pt]. Using Bolzano's theorem, prove that for all $\alpha \in (0, 1)$, the equation has a solution on $\mathbb{R}_{>0}$:

$$\ln(1 + x) = \alpha x$$

Solution. We look at the function $h(x) = \lim_{x \to 0} \frac{\ln(1+x)}{\alpha x}$, continuous on $(0, \infty)$, and search for points $x \in (0, \infty)$ such that $h(x) = 1$. Since $\lim_{x \to 0} h(x) = A > 1$ and $\lim_{x \to \infty} h(x) = B < 1$, Bolzano's theorem tells us that there exists some point $x \in (0, \infty)$ such that $h(x) = 1$. $\square$

3[2pt]. Prove that the above solution denoted by $p = p(\alpha)$ is unique.

Hint. Study the variation of the function $x \mapsto \ln(1 + x) - \alpha x$. $\square$

4[2pt]. Study the monotonicity of the function $p : \alpha \mapsto p(\alpha)$ for $\alpha \in (0, 1)$.

Hint. Let $\alpha < \alpha'$ and $x_\alpha, x_{\alpha'}$ be the resp. solutions. Look at the sign of $\ln(1 + \alpha) - \alpha' \cdot x_\alpha$. $\square$

2. Chebyshev polynomials[38pt]

In this exercise we look at the functions with $n \in \mathbb{Z}_{>0}$:

$$f_n(x) = \cos(n \arccos(x)).$$

Part I[8pt]. We study some basic properties of the function

$$\arccos : [-1, 1] \to [0, \pi].$$

1[3pt]. Recall why the function arccos is defined on $[-1, 1]$ and has value in $[0, \pi]$.

Solution. The function arccos is the inverse function of the function $\cos : [0, \pi] \to [-1, 1]$ restricted to the interval $[0, \pi]$. □

2[3pt]. Calculate $(\arccos)'(x)$ for $x \in (-1, 1)$ (recall that $\cos'(x) = -\sin(x)$ for all $x \in \mathbb{R}$).

Solution. A calculation yields $(\arccos)'(x) = -\frac{1}{\sqrt{1-x^2}}$ on $(-1, 1)$. For details, cf. lecture notes. □

3[2pt]. Calculate $\arccos(\cos(2\pi))$ and $\arccos(\cos(-\frac{\pi}{2}))$.

Solution. A direct way of doing it is using that $\cos(2\pi) = 1$ and $\arccos(1) = 0$ since $\cos(0) = 1$ and $0 \in [0, \pi]$. Similarly, $\cos(-\frac{\pi}{2}) = 0$ and $\arccos(0) = \frac{\pi}{2}$ since $\cos(\frac{\pi}{2}) = 0$ and $\frac{\pi}{2} \in [0, \pi]$.

There is a better (and more general) way, can you find it? □

Part II[8pt]. We study some basic properties of the function f_n for a fixed $n \in \mathbb{Z}_{>0}$.

1[2pt]. Show that the domain of definition $\mathcal{D}$ of f_n is independent of n.

Solution. Since cos is defined on $\mathbb{R}$, for f_n to be well-defined, we just need that $\arccos(x)$ is well-defined: this is independent of n. □

2[2pt]. Study the parity of f_n according to the parity of n.

Solution. We write

$$f_n(-x) = \cos(n \arccos(-x)) = \cos(n(\pi - \arccos(x))) = \cos(n\pi - n \arccos(x)) = (-1)^n \cos(n \arccos(x)).$$

This shows that f_n has the same parity as n (do a sanity check for $n = 0, 1$). □

3[2pt]. Prove that for all $x \in [-\pi, \pi]$ (you can first start with $x \in [0, \pi]$),

$$f_n(\cos(x)) = \cos(nx).$$

Solution. By a parity argument if suffices to show for $x \in [0, \pi]$. Let $x \in [0, \pi]$. Then

$$f_n(\cos(x)) = \cos(n \arccos(\cos(x))) = \cos(nx).$$

For $x \in [-\pi, 0]$, we can write, since cos is even,

$$f_n(\cos(x)) = f_n(\cos(-x)) = \cos(n(-x)) = \cos(nx).$$

□

4[2pt]. Can the above equation be extended to all $x \in \mathbb{R}$?

Solution. For a general $x \in \mathbb{R}$ we can always write it in the form $x = x_0 + 2k\pi$ with $k \in \mathbb{Z}$ and $x_0 \in [-\pi, \pi]$. Then

$$f_n(\cos(x)) = f_n(\cos(x_0)) = \cos(n(x_0)) = \cos(nx - 2kn\pi) = \cos(nx).$$

So I guess that the answer is yes? □

Part III[8pt]**.** Let $n \in \mathbb{Z}_{>0}$. We study the limit

$$u_n = \lim_{x\to 0} \frac{\cos(nx) - 1}{\cos(x) - 1}.$$

1[2pt]. Calculate the following limit (you can use l'Hôpital's rule)

$$\lim_{x\to 0} \frac{\cos(x) - 1}{x^2}.$$

Solution. Using l'Hôpital,

$$\lim_{x\to 0} \frac{\cos(x) - 1}{x^2} = \lim_{x\to 0} \frac{-\sin(x)}{2x} = -\frac{1}{2}.$$

□

2[2pt]. Use the previous question to calculate the limit u_n.

Solution. Write

$$\lim_{x\to 0} \frac{\cos(nx) - 1}{\cos(x) - 1} = \lim_{x\to 0} \frac{x^2}{\cos(x) - 1} \cdot \frac{\cos(nx) - 1}{(nx)^2} \cdot n^2 = n^2.$$

□

3[2pt]. Using $z = \arccos(x)$, justify that the function f_n is derivable at $x = 1$.

Solution. Using this change of variables we have

$$\frac{f_n(x) - f_n(1)}{x - 1} = \frac{f_n(\cos(z)) - f_n(\cos(0))}{\cos(z) - \cos(0)} = \frac{\cos(nz) - 1}{\cos(z) - 1}$$

by a question before. The previous question shows that the limit of the right hand side as z goes to 0 exists. It follows from the derivability of cos that by composition, f_n is derivable at $x = 1$. □

4[2pt]. Calculate $f_n'(1)$.

Solution. Question 2 above yields $f_n'(1) = n^2$. □

Part IV[14pt]**.** We study the relations between f_n for different values of $n \in \mathbb{Z}_{>0}$.

1[2pt]. Considering $e^{ix} = \cos(x) + i\sin(x)$, prove the formula

$$\forall \alpha, \beta \in \mathbb{R}; \quad 2\cos(\alpha)\cos(\beta) = \cos(\alpha + \beta) + \cos(\alpha - \beta)$$

Hint. Calculate the real part of $e^{i(\alpha+\beta)}$ in two different ways. □

2[6pt]. For all $x \in \mathcal{D}$, express the functions $f_0(x), f_1(x), \ldots, f_4(x)$ without using cos nor arccos.

Hint. You can use the same idea as in the previous question to calculate $\cos(nx)$ in terms of $\cos(x)$: for example

$$\cos(2x) = 2\cos^2(x) - 1.$$

Then it follows that $f_2(x) = \cos(2\arccos(x)) = 2\cos^2(\arccos(x)) - 1 = 2x^2 - 1$. □

$3^{[2pt]}$. Use the above formula to prove that for all $x \in \mathcal{D}$ and $n \in \mathbb{Z}_{>0}$,

$$f_{n+2}(x) = 2x f_{n+1}(x) - f_n(x).$$

Solution. Apply the identity in Question 1 to $\alpha = (n+1)\arccos(x)$ and $\beta = \arccos(x)$ and write

$$f_{n+2}(x) + f_n(x) = \cos((n+2)\arccos(x)) + \cos(n\arccos(x)) = 2\cos((n+1)\arccos(x))\cos(\arccos(x)).$$

The right hand side is nothing but $2x f_{n+1}(x)$. □

$4^{[2pt]}$. Solve the equations $f(x) = 0$ and $f'(x) = 0$.

Erratum. It should be solving $f_n(x) = 0$ and $f_n'(x) = 0$. This follows from a direct calculation. □

$5^{[2pt]}$. Show that for all $n \in \mathbb{Z}_{>0}$, f_n is a polynomial function.

Solution. This follows from Question 3 by induction. □