Algebra II

Exercise 1 (23.1.2020)

By doing exercises you can get bonus points for the exam as follows:
$30\% \to 1$, $40\% \to 2$, ... , $80\% \to 6$ points.

1. Give examples which show that the implications $f \circ g = f \circ h \Rightarrow g = h$ and $g \circ f = h \circ f \Rightarrow g = h$ don't hold for all functions.
Prove that $f \circ g = f \circ h \Rightarrow g = h$ holds, if f is injective.
Prove that $g \circ f = h \circ f \Rightarrow g = h$ holds, if f is surjective.

2. Let G be a group and $H \leq G$. Prove that the cosets of the subgroup H form a partition of the group G and that the following are equivalent:
1) $xH = yH$
2) $x^{-1}y \in H$
3) $x \in yH$
4) $x, y \in zH$ for some $z \in G$.
[Recall that if A is a set, then a collection of non-empty subsets of A is called a *partition* (ositus), if every element $a \in A$ belongs to exactly one of the sets in the collection.]

3. a) Consider the group $(\mathbb{Q}, +)$. Prove that the relation

$$a \sim b \Leftrightarrow a - b \in \mathbb{Z}$$

is an equivalence relation, which is compatible with the group operation $+$.
[Here "compatible" means the following: if $x \sim x'$ and $y \sim y'$, then $x + y \sim x' + y'$.]
b) Prove that the set of equivalence classes $\mathbb{Q}/\mathbb{Z}$ with the induced group operation is an infinite Abelian group, where the order of each element is finite.

4. a) For any given $n \in \mathbb{N}_+$, there exists the group $\mathbb{Z}_n$, which has order n. During the first lecture I mentioned that there exists essentially only one group of order 15. What about $\mathbb{Z}_5 \times \mathbb{Z}_3$?

b) Prove that every group with 4 elements is isomorphic either to the group $\mathbb{Z}_4$ or to the group $\mathbb{Z}_2 \times \mathbb{Z}_2$. [Hint: You can begin by considering possible orders of elements.]

5. Let $(R, +, \cdot)$ be a ring. Denote by 0 the neutral element with respect to $+$ and by 1 the neutral element with respect to $\cdot$.

a) Prove that $0 \cdot x = x \cdot 0 = 0$ for every $x \in R$.

b) Prove that if R has at least two elements, then $0 \neq 1$.

6. Let $m, n \in \mathbb{N}$ and $m \geq 1$. Let $R_{m,n}$ be the equivalence relation

$$xR_{m,n}y \Leftrightarrow x = y \text{ or } (x \geq n \text{ and } y \geq n \text{ and } m|x - y)$$

in the set $\mathbb{N}$. Prove that the addition in the monoid $(\mathbb{N}, +)$ and the relation $R_{m,n}$ are compatible and describe the quotient monoid $\mathbb{N}/R_{m,n}$.