FOURIER ANALYSIS I. (spring)

1. EXERCISES (Thursday 23.1 14-16 in room C123)

NOTE that these exercises deal just a little bit also with some of the material included in the Monday lecture 20.1. !

1. Compute the Fourier coefficients of the function $f(x) = |x|$, for $|x| \leq \pi$.

2. Compute the Fourier coefficients of the function $f(x) = cos(x/2)$, $x \in [-\pi, \pi]$.

3. Use Euler formulas $\sin(x) = \dfrac{e^{ix} - e^{-ix}}{2i}$ and $\cos(x) = \dfrac{e^{ix} + e^{-ix}}{2}$ to prove the orthogonality relations

$$\frac{1}{\pi}\int_0^{2\pi} \sin(nx)\sin(mx)dx = \delta_{n,m}, \qquad \frac{1}{\pi}\int_0^{2\pi} \cos(nx)\cos(mx)dx = \delta_{n,m}$$

and

$$\frac{1}{\pi}\int_0^{2\pi} \sin(nx)\cos(mx)dx = 0$$

for all integers $n, m \geq 1$.

4. Let f be a 2π-periodic function on $\mathbf{R}$ that is integrable on $[0, 2\pi]$. Show that

$$\int_a^{a+2\pi} f(x)dx$$

is independent of $a \in \mathbf{R}$.

5. (i) Assume that $f \in L^1(-\pi, \pi)$ is odd, i.e. $f(-x) = -f(x)$. Show that then the Fourier series of f is a pure sine series, i.e. can be expressed in terms of functions $\sin(nx)$, $n \in \mathbf{Z}$.

(ii) How do you characterise functions whose Fourier series is a pure cosine series?

6. How do you express the Fourier coefficients $\widehat{g}(n)$ assuming that you know those of f when f is 2π-periodic and

(i) $g(x) = f(\pi - x)$?

(ii) $g(x) = 1 - f(4x)$ for $x \in [-\pi, \pi]$?

7*[1] Try to find out Let$f$ be the 2π-periodic function defined by $f(x) = \cosh(x) = (e^x + e^{-x})/2$ for $|x| \leq \pi$ .Express it as a Fourier series. Assuming the convergence of the series (which we will prove on Monday!), compute

$$\sum_{k=1}^{\infty} \frac{1}{1+k^2}.$$

[1] These ∗-exercises are extras for afficinadoes, not required to get full points from exercises