FOURIER ANALYSIS I. (Spring 2020)

2. EXERCISES (Thursday 30.1 14-16 in room C123)

NOTE that these exercises may deal just a little bit also with some of the material included in the Monday lecture 27.1. !

1. Last week you were asked to compute the Fourier series of the function $f(x) = \cos(x/2)$. Show that the series converge to $f(x)$ at every point. What identity do you get from this when you substitute $x = 0$?

2. Let $f(x)$ and $g(x)$ be 2π-periodic functions such that $f,\, g \in L^1[-\pi, \pi]$. Show that the Fourier coefficients of the convolution $f * g$ are given by

$$\widehat{(f * g)}(n) = \widehat{f}(n)\,\widehat{g}(n), \quad n \in \mathbf{N}.$$

[Hint: You may use e.g. Fubini's theorem]

3. Which of the following families $\{K_n\}_{n=1}^{\infty}$ of 2π-periodic functions is a family of *good kernels*, which not:

$$\text{(i)}\ \ K_n(x) = 2\pi n \max\{0,\, 1 - n|x|\}, \quad \text{(ii)}\ \ K_n(x) = \frac{\sin(nx)}{x^2}, \quad \text{(iii)}\ \ K_n(x) = (n+1)\left(1 - \frac{|x|}{\pi}\right)^n.$$

The kernels $K_n(x)$ above are given on the interval $x \in (-\pi, \pi)$.

4. Let f be a continuous and g an integrable 2π-periodic function. Prove that then the convolution $f * g$ is continuous.

5. (i) Show that for every 2π-periodic function $f \in L^1[-\pi, \pi]$ we have

$$\widehat{f}(n) = \frac{1}{4\pi} \int_0^{2\pi} e^{-inx}\big(f(x) - f(x + \pi/n)\big)dx.$$

(ii) If $f \in C_{\#}(-\pi, \pi)$ is Hölder-continuous[1] with exponent $\alpha \in (0, 1]$, show that

$$|\widehat{f}(n)| \leq C|n|^{-\alpha}, \quad \text{for } |n| \geq 1.$$

6. From lectures we know that the Fourier coefficients $\widehat{f}(n)$ of a function $f \in C_{\#}^{(k)}$ decay to zero with the speed $O(|n|^{-k})$ as $n \to \infty$.

Show that a converse result holds in the following sense: If f is continuous on the interval $[-\pi, \pi]$ and for an integer $k \geq 2$ there is a constant $C = C_k$ for which

$$|\widehat{f}(n)| \leq C_k(1 + |n|)^{-k}, \quad n \in \mathbf{Z},$$

[1] i.e., $|f(x) - f(y)| \leq C|x - y|^{\alpha}$ with some constant C

then $f \in C_{\#}^{k-2}(-\pi, \pi)$.

[Hint: Use suitable results from the analysis course to verify that in the situation $k \geq 3$ one may differentiate the Fourier series of f term by term.]

7^{*2} Let $f : [0,1] \to \mathbf{R}$ be continuous. We define its 'moments' $M_f(n)$ for $n = 0, 1, 2, \ldots$ by the formula

$$M_f(n) := \int_0^1 x^n f(x) dx.$$

Show that the moments determine the funktion f uniquely, i.e. if $g : [0,1] \to \mathbf{R}$ is another continuous function and $M_g(n) = M_f(n)$ for all indices $n \geq 0$, then $f(x) = g(x)$ for all $x \in [0,1]$.

[HInt: try to copy the proof given in the lectures showing that a continuous function is uniquely determined by its Fourier coefficients.]

[2] These $*$-exercises are extras for afficinadoes, not required to get full points from exercises