Algebra II

Exercise 2 (30.1.2020)

1. a) Prove that every group homomorphism $f\colon (\mathbb{Z},+) \to (\mathbb{Z},+)$ is of the form $f(x) = n \cdot x$ for some $n \in \mathbb{Z}$.

b) Let $n \geq 2$. Prove that there doesn't exist other group homomorphisms $(\mathbb{Z}_n,+) \to (\mathbb{Z},+)$ than the zero homomorphism.

c) Prove that there exists a group homomorphism $g\colon \mathbb{Z}_n \to \mathbb{Z}_m$, for which $g([1]) = [1]$, if and only if m divides n. [Hint: Proposition 1.14]

2. Let X be a G-set.

a) Prove that the relation

$$x \sim x' \Leftrightarrow x = gx' \text{ for some } g \in G$$

is an equivalence relation in the set X, and the equivalence classes are the orbits. (Hence the orbits form a partition of X.)

b) Prove that the stabilizers of elements are subgroups of G, but not necessarily normal subgroups. [Hint: You can consider the symmetry group of a triangle.]

c) Prove that if $g \in G$, $x \in X$, then the stabilizer (sometimes called the isotropy subgroup) at gx is

$$G_{gx} = gG_xg^{-1}.$$

3. Let

$$G = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \;\middle|\; a, b \in \mathbb{R}, a \neq 0 \right\}.$$

Prove that equipped with matrix multiplication G is a group and

$$H = \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \;\middle|\; n \in \mathbb{Z} \right\}$$

is a subgroup of G. Which familiar group is H isomorphic with? Let

$$g = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \in G.$$

Prove that $g^{-1}Hg \subsetneq H$.

4. Prove the following version of Proposition 1.14: Suppose that $f: G \to H$ and $p: G \to G'$ are group homomorphisms and p is surjective. Then there exists a homomorphism $\bar{f}: G' \to H$, for which $f = \bar{f} \circ p$, if and only if $\mathrm{Ker}(p) \subset \mathrm{Ker}(f)$.

5. Suppose that G is a group. Prove the following basic facts related to conjugation:

a) If $g \in G$ and $H \leq G$, then $gHg^{-1} \leq G$.

b) The formula $(g, h) \mapsto ghg^{-1}$ defines an action of G on itself. The formula $(g, H) \mapsto gHg^{-1}$ defines an action of G in the set of all subgroups of G.

6. Continuation to exercise 6 of last week.
Prove that every non-trivial equivalence relation of the monoid $(\mathbb{N}, +)$, which is compatible with the addition, is the relation $R_{n,m}$ for some $n, m \in \mathbb{N}$, $m \geq 1$.

(The trivial relations are:
(i) $x \sim y$ for all $x, y \in \mathbb{N}$ and
(ii) $x \sim y \Leftrightarrow x = y$.)