Fourier Analysis I

Spring 2020

Homework 1

Exercise session: Thu 23 January, 14:15 - 16:00, Exactum C123; Stefanos Lappas, stefanos.lappas@helsinki.fi.

1. Compute the Fourier coefficients of the function $f(x) = |x|$, for $|x| \leq \pi$.

Proof. First of all,

$$\widehat{f}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x| dx = \frac{1}{\pi}\int_{0}^{\pi} x dx = \pi/2.$$

Let us now compute $\widehat{f}(n)$ for $n \neq 0$

$$\begin{aligned}\widehat{f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx}|x| dx \\ &= \frac{1}{2\pi}\int_{0}^{\pi} (e^{-inx} + e^{inx}) x dx \\ &= \frac{1}{\pi}\int_{0}^{\pi} \cos(nx) x dx.\end{aligned}$$

The above integral can be computed by standard integrations by parts. The antiderivative is

$$\int x\cos(nx)dx = \frac{x\sin(nx)}{n} - \int \frac{\sin(nx)}{n} dx = \frac{x\sin(nx)}{n} - \frac{-\cos(nx)}{n^2} = \frac{x\sin(nx)}{n} + \frac{\cos(nx)}{n^2}$$

Thus

$$\frac{1}{\pi}\int_{0}^{\pi} \cos(nx) x dx = \frac{1}{\pi}\frac{\pi n\sin(\pi n) - 1 + \cos(\pi n)}{n^2}.$$

For integer $n \neq 0$, we have

$$\frac{1}{\pi}\frac{\pi n\sin(\pi n) - 1 + \cos(\pi n)}{n^2} = \frac{-1 + \cos(\pi n)}{\pi n^2} = \begin{cases} 0, & \text{if } n \text{ is even} \\ -\frac{2}{\pi n^2}, & \text{if } n \text{ is odd} \end{cases}$$

So we have

$$\widehat{f}(n) = \begin{cases} \frac{\pi}{2}, & \text{if } n = 0 \\ 0, & \text{if } n \neq 0 \text{ is even} \\ -\frac{2}{\pi n^2}, & \text{if } n \text{ is odd} \end{cases}$$

□

2. Compute the Fourier coefficients of the function $f(x) = \cos(x/2), x \in [-\pi, \pi]$.

Proof. We compute the Fourier coefficients

$$
\begin{aligned}
\widehat{f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx} cos(x/2) dx \\
&= \frac{1}{4\pi}\int_{-\pi}^{\pi} e^{-inx}(e^{ix/2} + e^{-ix/2}) dx \\
&= \frac{1}{4\pi}\int_{-\pi}^{\pi} (e^{ix(1/2-n)} + e^{-ix(1/2+n)}) dx \\
&= \frac{1}{4\pi}\Big[\Big(\frac{1}{i(1/2-n)} e^{ix(1/2-n)} + \frac{1}{-i(1/2+n)} e^{-ix(1/2+n)}\Big)\Big]_{-\pi}^{\pi} \\
&= \frac{1}{2\pi i(1-2n)}(e^{i\pi(1/2-n)} - e^{-i\pi(1/2-n)}) + \frac{1}{-2\pi i(1+2n)}(e^{-i\pi(1/2+n)} - e^{i\pi(1/2+n)}) \\
&= \frac{1}{2\pi i(1-2n)}(ie^{-i\pi n} + ie^{i\pi n}) + \frac{1}{2\pi i(1+2n)}(ie^{-i\pi n} + ie^{i\pi n}) \\
&= \frac{1}{2\pi i(1-2n)} 2i(-1)^n + \frac{1}{2\pi i(1+2n)} 2i(-1)^n \\
&= \frac{(-1)^n}{\pi}\Big(\frac{1}{1-2n} + \frac{1}{1+2n}\Big) \\
&= \frac{2(-1)^n}{\pi(1-4n^2)}.
\end{aligned}
$$

Hence,

$$
\widehat{f}(n) = \frac{2(-1)^n}{\pi(1-4n^2)}.
$$

□

3. Use Euler formulas $\sin(x) = \frac{e^{ix}-e^{-ix}}{2i}$ and $\cos(x) = \frac{e^{ix}+e^{-ix}}{2}$ to prove the orthogonality relations

$$
\frac{1}{\pi}\int_0^{2\pi} \sin(nx)\sin(mx) dx = \delta_{n,m}, \qquad \frac{1}{\pi}\int_0^{2\pi} \cos(nx)\cos(mx) dx = \delta_{n,m}
$$

and

$$
\frac{1}{\pi}\int_0^{2\pi} \sin(nx)\cos(mx) dx = 0
$$

for all integers $n, m \geq 1$.

Proof. For the first orthogonal relation we have

$$\begin{aligned}\frac{1}{\pi}\int_0^{2\pi}\sin(nx)\sin(mx)dx &= \frac{1}{\pi}\int_0^{2\pi}\frac{e^{inx}-e^{-inx}}{2i}\frac{e^{imx}-e^{-imx}}{2i}dx\\ &= -\frac{1}{4\pi}\int_0^{2\pi}e^{ix(n+m)}-e^{ix(n-m)}-e^{-ix(n-m)}+e^{-ix(n+m)}dx. \quad (1)\end{aligned}$$

If $n=m$ then from (1) we have

$$\begin{aligned}-\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{ix2n}+e^{-ix2n}}{2}dx+1 &= -\frac{1}{2\pi}\int_0^{2\pi}\cos(2nx)dx+1\\ &= -\frac{1}{2\pi}\frac{\sin(4\pi n)}{2n}+1\\ &= 1.\end{aligned}$$

If $n\neq m$ then from (1) we have

$$\begin{aligned}-\frac{1}{2\pi}\int_0^{2\pi}\cos((n+m)x)dx &+ \frac{1}{2\pi}\int_0^{2\pi}\cos((n-m)x)dx\\ &= -\frac{1}{2\pi}\frac{\sin((n+m)2\pi)}{n+m}+\frac{1}{2\pi}\frac{\sin((n-m)2\pi)}{n-m}\\ &= 0.\end{aligned}$$

Hence,

$$\frac{1}{\pi}\int_0^{2\pi}\sin(nx)\sin(mx)dx = \delta_{n,m}.$$

Now, we focus on the second orthogonal relation

$$\begin{aligned}\frac{1}{\pi}\int_0^{2\pi}\cos(nx)\cos(mx)dx &= \frac{1}{\pi}\int_0^{2\pi}\frac{e^{inx}+e^{-inx}}{2}\frac{e^{imx}+e^{-imx}}{2}dx\\ &= \frac{1}{4\pi}\int_0^{2\pi}e^{ix(n+m)}+e^{ix(n-m)}+e^{-ix(n-m)}+e^{-ix(n+m)}dx. \quad (2)\end{aligned}$$

If $n=m$ then from (2) we have

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{ix2n}+e^{-ix2n}}{2}dx+1 &= \frac{1}{2\pi}\int_0^{2\pi}\cos(2nx)dx+1\\ &= \frac{1}{2\pi}\frac{\sin(4\pi n)}{2n}+1\\ &= 1.\end{aligned}$$

If $n \neq m$ then from (2) we have

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\cos((n+m)x)dx + \frac{1}{2\pi}\int_0^{2\pi}\cos((n-m)x)dx \\ &= \frac{1}{2\pi}\frac{\sin((n+m)2\pi)}{n+m} + \frac{1}{2\pi}\frac{\sin((n-m)2\pi)}{n-m} \\ &= 0.\end{aligned}$$

Hence,

$$\frac{1}{\pi}\int_0^{2\pi}\cos(nx)\cos(mx)dx = \delta_{n,m}.$$

For the last orthogonal relation we have

$$\begin{aligned}\frac{1}{\pi}\int_0^{2\pi}\sin(nx)\cos(mx)dx &= \frac{1}{\pi}\int_0^{2\pi}\frac{e^{inx}-e^{-inx}}{2i}\frac{e^{imx}+e^{-imx}}{2}dx \\ &= \frac{1}{4\pi i}\int_0^{2\pi} e^{ix(n+m)} + e^{ix(n-m)} - e^{-ix(n-m)} - e^{-ix(n+m)}dx \\ &= \frac{1}{2\pi}\int_0^{2\pi}\Big(\frac{e^{ix(n+m)}-e^{-ix(n+m)}}{2i} + \frac{e^{ix(n-m)}-e^{-ix(n+m)}}{2i}\Big)dx \\ &= \frac{1}{2\pi}\int_0^{2\pi}\sin((n+m)x)dx + \frac{1}{2\pi}\int_0^{2\pi}\sin((n-m)x)dx \\ &= -\frac{1}{2\pi}\frac{\cos((n+m)2\pi)}{n+m} + \frac{1}{2\pi}\frac{1}{n+m} - \frac{1}{2\pi}\frac{\cos((n-m)2\pi)}{n-m} \\ &+ \frac{1}{2\pi}\frac{1}{n-m} \\ &= 0.\end{aligned}$$

Hence,

$$\frac{1}{\pi}\int_0^{2\pi}\cos(nx)\cos(mx)dx = 0.$$

□

4. Let f be a 2π-periodic function on $\mathbf{R}$ that is integrable on $[0, 2\pi]$. Show that

$$\int_a^{a+2\pi} f(x)dx$$

is independent of $a \in \mathbf{R}$.

Proof. Using the fact that f is 2π-periodic function we have that

$$\begin{aligned}\int_a^{a+2\pi} f(x)\,dx - \int_0^{2\pi} f(x)\,dx &= \left(\int_a^{2\pi} f(x)\,dx + \int_{2\pi}^{a+2\pi} f(x)\,dx\right)\\ &- \left(\int_0^{a} f(x)\,dx + \int_a^{2\pi} f(x)\,dx\right)\\ &= \int_{2\pi}^{a+2\pi} f(x)\,dx - \int_0^a f(x)\,dx\\ &= \int_0^a f(x+2\pi)\,dx - \int_0^a f(x)\,dx\\ &= \int_0^a (f(x+2\pi) - f(x))\,dx\\ &= \int_0^a 0\,dx\\ &= 0\end{aligned}$$

for any $a \in \mathbf{R}$.

□

5. (i) Assume that $f \in L^1(-\pi,\pi)$ is odd, i.e. $f(-x) = -f(x)$. Show that then the Fourier series of f is a pure sine series, i.e. can be expressed in terms of functions $\sin(nx), n \in \mathbf{Z}$.

(ii) How do you characterise functions whose Fourier series is a pure cosine series?

Proof. (i) Suppose f is odd. We use a substitution $t = -x$ to compute that

$$\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx} f(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{int} f(-t)dt = -\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(-nt)} f(t)dt = -\widehat{f}(-n).$$

This computation also shows that $\widehat{f}(0) = -\widehat{f}(0)$, so $\widehat{f}(0) = 0$. Let us now show that the Fourier series of f consists only of sine functions. As

$$\sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix})$$

we may represent the Fourier series as

$$\sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{inx} = \sum_{n=1}^{\infty} \left(\widehat{f}(n)e^{inx} + \widehat{f}(-n)e^{-inx}\right)$$
$$= \sum_{n=1}^{\infty} \widehat{f}(n)(e^{inx} - e^{-inx})$$
$$= \sum_{n=1}^{\infty} 2i\widehat{f}(n)\sin(nx).$$

(ii) Suppose f is even. We use a substitution $t = -x$ to compute that

$$\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx} f(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{int} f(-t)dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(-nt)} f(t)dt = \widehat{f}(-n).$$

Let us now show that the Fourier series of f consists only of cosine functions. As

$$\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix})$$

we may represent the Fourier series as

$$\sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{inx} = \widehat{f}(0) + \sum_{n=1}^{\infty} \left(\widehat{f}(n)e^{inx} + \widehat{f}(-n)e^{-inx}\right)$$
$$= \widehat{f}(0) + \sum_{n=1}^{\infty} \widehat{f}(n)(e^{inx} + e^{-inx})$$
$$= \widehat{f}(0) + \sum_{n=1}^{\infty} 2\widehat{f}(n)\cos(nx).$$

□

6. How do you express the Fourier coefficients of $\widehat{g}(n)$ assuming that you know those of f when f is 2π-periodic and

(i) $g(x) = f(\pi - x)$?

(ii) $g(x) = 1 - f(4x)$ for $x \in [-\pi, \pi]$?

Proof. **(i)** Using substitution $y = \pi - x$ shows that

$$\begin{aligned}\widehat{g}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)e^{-inx}dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\pi - x)e^{-inx}dx \\ &= \frac{1}{2\pi}\int_{0}^{2\pi} f(y)e^{-in(\pi-y)}dy \\ &= e^{-in\pi}\frac{1}{2\pi}\int_{0}^{2\pi} f(y)e^{iny}dy \\ &= e^{-in\pi}\widehat{f}(-n).\end{aligned}$$

(ii) We will prove that

$$\widehat{g}(n) = \begin{cases} 1 - \widehat{f}(0), & \text{if } n = 0 \\ 0, & \text{if } n \neq 4k, k \in \mathbb{Z} \\ -\widehat{f}(n/4), & \text{if } n = 4k, k \in \mathbb{Z} \end{cases}$$

First of all,

$$\widehat{g}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(1 - f(4x))dx = 1 - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(4x)dx = 1 - \widehat{f}(0).$$

Let us now compute $\widehat{f}(n)$ for $n \neq 0$. We make a substitution $y = 4x$ and use the fact that f is 2π-periodic to obtain

$$\begin{aligned}\widehat{g}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(1 - f(4x))e^{-inx}dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx}dx - \frac{1}{2\pi}\int_{0}^{2\pi} f(4x)e^{-inx}dx \\ &= -\frac{1}{2\pi in}(e^{-in\pi} - e^{in\pi}) - \frac{1}{8\pi}\int_{0}^{8\pi} f(y)e^{-i(n/4)y}dy \\ &= \frac{1}{\pi n}\sin(n\pi) - \frac{1}{8\pi}\Big(\int_{0}^{2\pi} f(y)e^{-i(n/4)y}dy + \int_{2\pi}^{4\pi} f(y)e^{-i(n/4)y}dy \\ &+ \int_{4\pi}^{6\pi} f(y)e^{-i(n/4)y}dy + \int_{6\pi}^{8\pi} f(y)e^{-i(n/4)y}dy\Big)\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{8\pi}\Big(\int_0^{2\pi} f(y)e^{-i(n/4)y}dy + \int_0^{2\pi} f(y+2\pi)e^{-i(n/4)(y+2\pi)}dy \\
&+ \int_0^{2\pi} f(y+4\pi)e^{-i(n/4)(y+4\pi)}dy + \int_0^{2\pi} f(y+6\pi)e^{-i(n/4)(y+6\pi)}dy\Big) \\
&= -\frac{1}{8\pi}\int_0^{2\pi} f(y)\Big(e^{-i(n/4)y} + e^{-i(n/4)(y+2\pi)} + e^{-i(n/4)(y+4\pi)} + e^{-i(n/4)(y+6\pi)}\Big)dy \\
&= -\frac{1}{8\pi}\int_0^{2\pi} f(y)e^{-i(n/4)y}\Big(1 + e^{-i(n/2)\pi} + e^{-in\pi} + e^{-i(3n/2)\pi}\Big)dy \\
&= \begin{cases} 0, & \text{if } n \neq 4k, k \in \mathbb{Z} \\ -\widehat{f}(n/4), & \text{if } n = 4k, k \in \mathbb{Z}\ . \end{cases}
\end{aligned}
$$

□

Remark. *We have in fact a more general result: if k is a positive integer and $g(x) = 1 - f(kx)$, then*

$$
\widehat{g}(n) = \begin{cases} 1 - \widehat{f}(0), & \text{if } n = 0 \\ 0, & \text{if } k \text{ does not divide } n \\ -\widehat{f}(n/k), & \text{if } k \text{ divides } n \end{cases}
$$

The essential observation when proving this is that $\sum_{j=0}^{p-1} e^{-i(j/p)\pi} = 0$ for any integer $p \geq 2$, the details are left for an interested reader.

7. Let f be the 2π-periodic function defined by $f(x) = \coth(x) = (e^x + e^{-x})/2$ for $|x| \leq \pi$. Express it as a Fourier series. Assuming the convergence of the series, compute

$$
\sum_{k=1}^{\infty} \frac{1}{1+k^2}.
$$

Proof. First of all,

$$
\widehat{f}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \coth(x)dx = \frac{1}{2\pi}(\sinh(\pi) - \sinh(-\pi)) = \frac{\sinh(\pi)}{\pi}
$$

Let us now compute $\widehat{f}(k)$ for $k \neq 0$

$$\begin{aligned}
\widehat{f}(k) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \coth(x) e^{-ikx} dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Big(\frac{e^{x}+e^{-x}}{2}\Big) e^{-ikx} dx \\
&= \frac{1}{4\pi}\int_{-\pi}^{\pi} \Big(e^{x(1-ik)} + e^{-x(1+ik)}\Big) dx \\
&= \frac{1}{4\pi}\Big(\frac{1}{1-ik}\Big[e^{\pi(1-ik)} - e^{-\pi(1-ik)}\Big] + \frac{1}{1+ik}\Big[e^{\pi(1+ik)} - e^{-\pi(1+ik)}\Big]\Big) \\
&= \frac{(-1)^k}{2\pi(1-ik)}\frac{e^{\pi}-e^{-\pi}}{2} + \frac{(-1)^k}{2\pi(1+ik)}\frac{e^{\pi}-e^{-\pi}}{2} \\
&= (-1)^k \sinh(\pi)\Big(\frac{1}{2\pi(1-ik)} + \frac{1}{2\pi(1+ik)}\Big) \\
&= \frac{(-1)^k \sinh(\pi)}{\pi(1+k^2)}.
\end{aligned}$$

Thus, assuming the convergence of the Fourier series of $f(x) = \coth(x)$ and by (ii) of the Exercise 5 we find that

$$\coth(x) = \frac{\sinh(\pi)}{\pi}\Big(1 + 2\sum_{k=1}^{\infty}\frac{(-1)^k \cos(kx)}{1+k^2}\Big) \quad \text{for all} \quad x \in [-\pi, \pi] \tag{3}$$

Choosing $x = \pi$ in (3) we get,

$$\sum_{k=1}^{\infty}\frac{1}{1+k^2} = \frac{\pi\coth(\pi) - 1}{2}.$$

$\square$