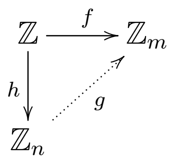
Algebra II. Exercise 2.
Solutions.

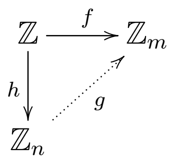
1. a) Let $f\colon (\mathbb{Z},+) \to (\mathbb{Z},+)$ be a group homomorphism. Denote $n = f(1)$. Then $f(2) = f(1+1) = f(1) + f(1) = 2n$, $f(3) = 3n$ etc. Thus $f(x) = nx$ for all $x > 0$. Furthermore $f(0) = 0$ and $f(-x) = -f(x) = -nx = n \cdot (-x)$, when $x > 0$. Thus $f(x) = nx$ for all $x \in \mathbb{Z}$.
b) Suppose the contrary that $f\colon \mathbb{Z}_n \to \mathbb{Z}$ is a homomorphism and $f([x]) = a \neq 0$ for some $[x] \in \mathbb{Z}_n$. Since in the group $\mathbb{Z}_n$ we have $n \cdot [x] = [nx] = [0]$ and f is a homomorphism, then

$$f([0]) = f(n \cdot [x]) = n \cdot f([x]) = na \neq 0,$$

which is a contradiction, since for a homomorphism we always have $f([0]) = 0$.
c) Consider the diagram



where $h(x) = [x]_n$ and $f(x) = [x]_m$, $x \in \mathbb{Z}$. Especially $h(1) = [1]_n$ and $f(1) = [1]_m$. The subgroup N appearing in Proposition 1.14 is here $n\mathbb{Z}$ and $\mathrm{Ker}(f) = m\mathbb{Z}$. Observe that $n\mathbb{Z} \subset m\mathbb{Z}$, if and only if m divides n.
"$\Rightarrow$": If $m|n$, we have $n\mathbb{Z} \subset m\mathbb{Z}$, and by Proposition 1.14 there exists a homomorphism $g\colon \mathbb{Z}_n \to \mathbb{Z}_m$, such that the diagram commutes; especially $g([1]_n) = [1]_m$.
"$\Leftarrow$": Suppose that $g\colon \mathbb{Z}_n \to \mathbb{Z}_m$ is a group homomorphism, for which $g([1]_n) = [1]_m$. The homomorphism condition gives us $g([2]_n) = [2]_m$, $g([3]_n) = [3]_m$, $g([-1]_n) = [-1]_m$ etc., and hence $g([x]_n) = [x]_m$ for every $x \in \mathbb{Z}$. Thus the diagram



commutes, so by Proposition 1.14 we have $N \subset \text{Ker}(f)$, that is, $n\mathbb{Z} \subset m\mathbb{Z}$ and thus $m|n$.

2. a)

- $x \sim x$, since $x = ex$.
- $x \sim y \Rightarrow x = gy$ for some $g \Rightarrow g^{-1}x = g^{-1}gy = y \Rightarrow y = g^{-1}x \Rightarrow y \sim x$.
- $x \sim y$ ja $y \sim z \Rightarrow x = gy, y = hz$ for some $h, g \Rightarrow x = gy = g(hz) = (gh)z \Rightarrow x \sim z$.
- $y \in [x] \iff y = gx$ for some $g \in G \iff y \in Gx$.

b) Consider the stabilizer G_x of an element $x \in X$.

- $e \in G_x$, because $ex = x$.
- If $g, h \in G_x$, then $(gh)x = g(hx) = gx = x$, and thus $gh \in G_x$.
- If $g \in G_x$, then $gx = x$, from which we get $g^{-1}gx = g^{-1}x$, that is, $x = g^{-1}x$. Thus $g^{-1} \in G_x$.

Thus G_x is a subgroup.
Consider the symmetry group of the triangle, which is isomorphic with S_3. Label the vertices of the triangle by 1,2,3; then the stabilizer of the vertex 1 is $H = \{(1), (2\ 3)\}$. This is not a normal subgroup of S_3, which can be seen by choosing for example $g = (1\ 3) \in S_3$ and $h = (2\ 3) \in H$ and noticing that $ghg^{-1} = ... = (1\ 2) \notin H$.
Another, more general, example can be given as follows. Let G be a group and $H \leq G$, H not a normal subgroup. Consider the action

$$G \times G/H \to G/H$$

$$(\bar{g}, gH) \mapsto (\bar{g}g)H$$

and the element $eH \in G/H$. Now $\bar{g}(eH) = eH \iff \bar{g}H = eH \iff \bar{g} \in H$, so the stabilizer of the element eH is H. Thus a stabilizer is not always a normal subgroup.
c) "$\subset$": If $h \in G_{gx}$, we have $hgx = gx$, that is $g^{-1}hgx = x$. From this we see that $g^{-1}hg \in G_x$, and hence $h \in gG_xg^{-1}$.

"$\supset$": If $h \in gG_xg^{-1}$, then $h = g\bar{h}g^{-1}$ for some $\bar{h} \in G_x$. Now

$$h(gx) = g\bar{h}g^{-1}(gx) = g\bar{h}x = gx,$$

so $h \in G_{gx}$ (in the last step we used the fact that $\bar{h} \in G_x$).

3. We prove first, that G is a subgroup of the group $Gl(2, \mathbb{R})$:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in G,$$

$$\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a' & b' \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} aa' & ab' + b \\ 0 & 1 \end{bmatrix} \in G$$

(Observe that $aa' \neq 0$, because $a \neq 0$ and $a' \neq 0$) and

$$\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} a^{-1} & -ba^{-1} \\ 0 & 1 \end{bmatrix} \in G,$$

from which the claim follows.
Then we prove that $H \leq G$:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in H,$$

$$\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & m+n \\ 0 & 1 \end{bmatrix} \in H$$

and

$$\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -n \\ 0 & 1 \end{bmatrix} \in H,$$

from which the claim follows.
Let $f \colon \mathbb{Z} \to H$ be the function

$$n \mapsto \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}.$$

Clearly f is bijective. Furthermore,

$$f(n) \cdot f(m) = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & n+m \\ 0 & 1 \end{bmatrix} = f(n+m),$$

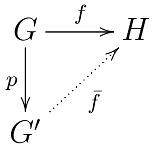
and thus f is a homomorphism, that is, an isomorphism $(\mathbb{Z}, +) \to (H, \cdot)$.
Because

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2n \\ 0 & 1 \end{bmatrix},$$

we see, that

$$g^{-1}Hg = \left\{ \begin{bmatrix} 1 & 2n \\ 0 & 1 \end{bmatrix} \,\middle|\, n \in \mathbb{Z} \right\} \subsetneq H.$$

4.



"$\Leftarrow$": Let $g' \in G'$. Because p is surjective, we may choose $g \in G$, such that $p(g) = g'$. Define

$$\bar{f}(g') := f(g).$$

$\bar{f}$ is well defined: If $\bar{g} \in G$ is another element, for which $p(\bar{g}) = g'$, then

$$p(g^{-1}\bar{g}) = p(g)^{-1}p(\bar{g}) = (g')^{-1}g' = e_{G'},$$

and thus $g^{-1}\bar{g} \in \mathrm{Ker}(p) \subset \mathrm{Ker}(f)$. Thus

$$f(g)^{-1}f(\bar{g}) = f(g^{-1}\bar{g}) = e_H,$$

and we have that $f(g) = f(\bar{g})$.
$\bar{f}$ is a homomorphism: Let $g_1', g_2' \in G'$; choose $g_1, g_2 \in G$ such that $p(g_i) = g_i',\ i = 1, 2$. Now

$$\bar{f}(g_1'g_2') = \bar{f}(p(g_1)p(g_2)) = \bar{f}(p(g_1g_2)) = f(g_1g_2) = f(g_1)f(g_2) = \bar{f}(g_1')\bar{f}(g_2').$$

The condition $f = \bar{f} \circ p$ follows directly from the above definition of the function $\bar{f}$.
"$\Rightarrow$": Suppose that $\mathrm{Ker}(p) \not\subset \mathrm{Ker}(f)$, that is, there exists $g \in G$, such that $p(g) = e_{G'}$ and $f(g) \neq e_H$. If $\bar{f}$ would exist, then we would have $\bar{f}(e_{G'}) = f(g) \neq e_H$, which is a contradiction, since $\bar{f}$ is a homomorphism.

5. a)

- $e = geg^{-1} \in gHg^{-1}$.
- If $ghg^{-1}, g\bar{h}g^{-1} \in gHg^{-1}$, then $(ghg^{-1})(g\bar{h}g^{-1}) = g(h\bar{h})g^{-1} \in gHg^{-1}$.
- If $ghg^{-1} \in gHg^{-1}$, then $(ghg^{-1})^{-1} = gh^{-1}g^{-1} \in gHg^{-1}$.

b) We prove that $\varphi: G \times G \to G,\ (g,h) \mapsto ghg^{-1}$ is an action of G:

- Clearly $\varphi(g,h) \in G$ for every $g,h \in G$.
- $\varphi(e,h) = ehe^{-1} = h$.
- $\varphi(g_1g_2,h) = (g_1g_2)h(g_1g_2)^{-1} = g_1g_2hg_2^{-1}g_1^{-1} = g_1(g_2hg_2^{-1})g_1^{-1}$
  $= g_1(\varphi(g_2,h))g_1^{-1} = \varphi(g_1,\varphi(g_2,h))$.

Denote $\mathcal{G}$ is the set of all subgroups of G. We prove that the function $\varphi: G \times \mathcal{G} \to \mathcal{G},\ (g,H) \mapsto gHg^{-1}$ defines an action of G:

- The function is well defined, since by item a) we have $gHg^{-1} \in \mathcal{G}$.
- $\varphi(e,H) = eHe^{-1} = H$.
- $\varphi(g_1g_2,H) = (g_1g_2)H(g_1g_2)^{-1} = g_1g_2Hg_2^{-1}g_1^{-1} = g_1(g_2Hg_2^{-1})g_1^{-1}$
  $= g_1(\varphi(g_2,H))g_1^{-1} = \varphi(g_1,\varphi(g_2,H))$.

6. Let $\sim$ be a non-trivial equivalence relation in the set $\mathbb{N}$, which is compatible with addition. Because $\sim$ is non-trivial, we can find $a,b \in \mathbb{N}$, $a \neq b$ and $a \sim b$. Let n be the smallest number, which is in the relation with also some other number than itself. Observe that we may have $n = 0$. Then every number $0, ..., n-1$ is in the relation only with itself, that is, the equivalence classes are $\{0\}, ..., \{n-1\}$.
We now have that the number n is in the relation with some number greater than n, but not with any number smaller than n. Then choose $m \in \{1,2,...\}$ such, that $n+m$ is the smallest number (which is greater than n), and which is in the relation with n. Thus $n \sim n+m$. From compatibility it now follows that $n+m \sim (n+m)+m = n+2m$, $n+2m \sim (n+m)+2m = n+3m$ etc. Thus

$$n \sim n + k \cdot m \quad \text{for every} \quad k \in \{0,1,2,...\}.$$

Analogously $n+1 \sim n+m+1$, $n+m+1 \sim n+2m+1$ etc., that is,

$$n+1 \sim n+1+k \cdot m \text{ for every } k \in \{0,1,2,...\}.$$

We obtain a partition

$$\{n, n+m, n+2m, ...\}$$

$$\{n+1, n+m+1, n+2m+1, ...\}$$

$$\vdots$$

$$\{n+m-1, n+2m-1, n+3m-1, ...\}.$$

Finally we prove that none of the numbers $n, n+1, ..., n+m-1$ are in the relation with each other. By the choice of the number m we have that n is not in the relation with any of the numbers $n+1, ..., n+m-1$. Assume the contrary, that there exist i, j, for which $n < i < j \leq n+m-1$, and $i \sim j$. Observe that we have $0 < j-i < m$. Now

$$n \sim n+m = i+(n+m-i) \sim j+(n+m-i) = (n+j-i)+m \sim n+j-i \leq n+m-1.$$

This is a contradiction with the choice of the number m.
Thus the equivalence classes are exactly the above mentioned, that is, $\sim$ is the relation $R_{m,n}$.
Remark. Actually the trivial relation [$x \sim y$ for all x, y] is also of the from $R_{m,n}$, namely $R_{1,0}$.