Algebra II

Exercise 3 (6.2.2020)

1. a) If X is a finite set, how many elements are there in the set $\mathrm{Sym}(X)$?
b) Give a condition concerning the number of elements of the sets G and X, such that when this condition holds, every action of G on X has the following property:

$$\exists\, g, g' \in G,\ g \neq g' \text{ for which } gx = g'x\ \forall x \in X.$$

Prove that then there also exists $\bar{g} \in G$, $\bar{g} \neq e$, such that $\bar{g}x = x$ for every $x \in X$.

2. Let X be a G-set, $x \in X$, $Y \subset X$. We know that the stabilizer G_x of an element is a subgroup of G. Prove that the stabilizer G_Y of a subset is always a submonoid of G. Give an example showing that G_Y is not necessarily a subgroup. Prove that G_Y is a subgroup, if Y or G_Y is a finite set.

3. Let

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 3 & 2 & 7 & 4 & 6 & 1 \end{pmatrix} \in S_7.$$

Represent τ as a product of disjoint cycles. Represent τ as a product of transpositions. Calculate $\mathrm{sgn}(\tau)$. Let $\sigma = (1\ 3)(5\ 2\ 6)$. Calculate $\sigma\tau\sigma^{-1}$.

4. a) Let i, j, k, l be (different) positive integers. Prove that

$$(ij)(kl) = (ilk)(ijk)$$

and

$$(ij)(ik) = (ikj).$$

b) Let $n \geq 3$. Prove that the subgroup of S_n generated by the 3-cycles is the alternating group A_n.

5. a) Let H be a subgroup of a group G. Prove that the following conditions are equivalent:

- H is a normal subgroup of G.
- For all $x, y \in G$: If $xH = yH$, then $x^{-1}H = y^{-1}H$.

b) Prove that if the index $[G : H]$ of H is 2, then H is a normal subgroup of G.

6. Prove the Cayley theorem: Every group G is isomorphic to a permutation group (that is, a subgroup of some symmetric group $\mathrm{Sym}(X)$). [Hint: Consider the action of G on itself defined by the formula $(g, x) \mapsto gx$.]
From this it especially follows that every finite group is isomorphic with a subgroup of some symmetric group S_n.