FOURIER ANALYSIS I. (Spring 2020)

3. EXERCISES (Thursday 6.2 14-16 in room C123)

1. **(i)** Let $N \in \mathbf{N}$. Show that there exists a non-trivial function $f \in L^1[-\pi, \pi]$ such that $F_N * f(x) = 0$ for all x.

(ii) Is there a non-trivial function $f \in L^1[-\pi, \pi]$ so that $F_N * f(x) = 0$ for all x and for *all* $N \geq 0$?

[Hint: both statements follows almost immediately from a suitable result established at the lectures.]

2. **(i)** Show that if there exist the limit $A := \lim_{n\to\infty} a_n$, then also

$$\lim_{N\to\infty} \frac{a_0 + a_1 + \ldots + a_{n-1}}{N} = A$$

(ii) Use part (i) to verify that if the series $\sum_{n=0}^{\infty} b_n$ converges and has sum S, then it is also Cesaro summable, i.e. if $s_n := \sum_{k=0}^{n} b_n$, we have

$$S = \lim_{N\to\infty} \frac{s_0 + s_1 + \ldots + s_{n-1}}{N}.$$

Show by a counter example that the converse is not true.

3. During the lectures it was shown that trigonometric polynomials are dense in $L^p(-\pi, \pi)$ if $1 \leq p < \infty$. Is the same result true if $p = \infty$?

4. Define $f(x) = 0$ for $x \in [-\pi, 0]$, $f(x) = \pi - x$ for $x \in [0, \pi)$, and extend f to 2π-perodic function. Compute the Fourier series of f. In which points does the Fourier series of the function $f(x)$ converge and to what value?

5. Provide more details to the proof of the result sketched at the lectures: Corollary 4.8; that is, show that if a 2π-periodic function $f(x)$ is piecewise [1] C^1 , then its Fourier series converges at every point, and

$$\lim_{N\to\infty} S_N f(x) = \lim_{t\to 0} \frac{f(x+t) + f(x-t)}{2}, \qquad x \in [-\pi, \pi].$$

6. Use the results of lectures so far to prove rigorously that every function $f : [0, \pi] \to \mathbf{C}$ that is Lipschitz-continuous (i.e. $|f(x) - f(y)| \leq C|x - y|$ for some $C < \infty$) and satisfies $f(0) = f(\pi) = 0$ can at each point $x \in [0, \pi]$ be expressed as a convergent sine series

$$f(x) = \sum_{k=1}^{\infty} c_k \sin(kx).$$

[1] i.e., apart from finitely many points $\{x_1, x_2, \ldots, x_n\} \subset [-\pi, \pi)$ the derivative $f'(x)$ exists and is continuous at x, and the derivative (and hence the function itself) has left and right limits at x_j for all $j = 1, \ldots n$.

Find an expression for the coefficients of c_k.

[Hint: extending f to an odd 2π-periodic function might be helpful...]

7^{*2} Can you find an example of a continuous function $f \in C_\sharp(-\pi,\pi)$ such that $S_n f(0) \to \infty$ as $n \to \infty$. An easier question: can you find an example of an integrable function $f \in L^1(-\pi,\pi)$ such that $S_n f(0) \to \infty$ as $n \to \infty$?

[2] These $*$-exercises are extras for afficinadoes, not required to get full points from exercises