Fourier Analysis I

Spring 2020

Homework 2

Exercise session: Thu 30 January, 14:15 - 16:00, Exactum C123; Stefanos Lappas, stefanos.lappas@helsinki.fi.

1. Last week you were asked to compute the Fourier series of the function $f(x) = \cos(x/2)$. Show that the series converge to $f(x)$ at every point. What identity do you get from this when you substitute $x = 0$?

Proof. By exercise sheet 1 the Fourier coefficients of f are:

$$\widehat{f}(n) = \frac{2(-1)^n}{\pi(1-4n^2)}$$

These Fourier coefficients converge quickly enough to zero as $n \to \infty$. The Weierstrass criterion implies that the Fourier series of f is absolutely summable. Since f is also continuous, we can deduce by Theorem 2.8 that the Fourier series converges uniformly to f. At $x = 0$ we have the identity

$$\sum_{n=-\infty}^{\infty} \frac{2(-1)^n}{\pi(1-4n^2)} = \cos(0/2) = 1.$$

As a curiosity, one could also deduce from this that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{1-4n^2} = \frac{\pi-2}{4}.$$

□

2. Let $f(x)$ and $g(x)$ be 2π-periodic functions such that $f, g \in L^1[-\pi, \pi]$. Show that the Fourier coefficients of the convolution $f * g$ are given by

$$(\widehat{f * g})(n) = \widehat{f}(n)\widehat{g}(n), \quad n \in \mathbf{N}.$$

Proof. Using Fubini's theorem (since $f, g \in L^1[-\pi,\pi]$) and making a substitution $x = y+t$ we obtain

$$\begin{aligned}
\widehat{(f*g)}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(f*g)(x)e^{-inx}dx &= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\Big(\int_{-\pi}^{\pi}f(x-y)g(y)dy\Big)e^{-inx}dx \\
&= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}g(y)\Big(\int_{-\pi}^{\pi}f(x-y)e^{-inx}dx\Big)dy \\
&= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}g(y)\Big(\int_{-\pi-y}^{\pi-y}f(t)e^{-in(y+t)}dt\Big)dy \\
&= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}g(y)e^{-iny}\Big(\int_{-\pi}^{\pi}f(t)e^{-int}\,dt\Big)dy \qquad (1)
\end{aligned}$$

On the right hand side of (1) the inner integral is independent of y and essentially $= \widehat{f}(n)$. We therefore can write

$$\widehat{(f*g)}(n) = \widehat{f}(n)\frac{1}{2\pi}\int_{-\pi}^{\pi}g(y)e^{-iny}\,dy = \widehat{f}(n)\widehat{g}(n).$$

□

3. Which of the following families $\{K_n\}_{n=1}^{\infty}$ of 2π-periodic functions is a family of *good kernels,* and which not:

(i) $K_n(x) = 2\pi n \max\{0, 1-n|x|\}$,

(ii) $K_n(x) = \frac{\sin(nx)}{x^2}$,

(iii) $K_n(x) = (n+1)(1-\frac{|x|}{\pi})^n$.

The kernels $K_n(x)$ above are given on the interval $x \in (-\pi,\pi)$.

Proof. First of all we recall the definition of the family of good kernels. A family $K_n \in L^1(-\pi,\pi), n \in \mathbf{N}$, is a good kernel if:

(a) $\frac{1}{2\pi}\int_{-\pi}^{\pi}K_n(x)dx = 1, \quad \forall n$

(b) $\frac{1}{2\pi}\int_{-\pi}^{\pi}|K_n(x)|dx \le C < \infty, \quad \forall n$

(c) For every $\delta > 0$ we have $\lim_{n\to\infty}\int_{\delta\le|x|\le\pi}|K_n(x)|dx = 0$.

(i) For the first kernel we check the above conditions:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}K_n(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}2\pi n\max\{0,1-n|x|\}dx = 2n\int_0^{1/n}(1-nx)dx = 1, \quad \forall n$$

and $\quad \frac{1}{2\pi}\int_{-\pi}^{\pi}|K_n(x)|dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi n \max\{0, 1-n|x|\}dx = 1 \le c, \quad \forall n \in \mathbf{N}.$

Now, fix $\delta > 0$ and choose $N \in \mathbf{N}$ such that $N^{-1} < \delta \implies 1 - n\delta < 0$ for $n \ge N$. Then

$$\int_{\delta\le|x|\le\pi}|K_n(x)|dx = \int_{\delta\le|x|\le\pi} 2\pi n \max\{0, 1-n|x|\}dx = 0, \quad n \ge N.$$

Thus, all the conditions are satisfied and $K_n(x) = 2\pi n \max\{0, 1-n|x|\}$ is a family of good kernels.

(ii) For the second kernel as sin is a smooth function we can choose some $A, \epsilon > 0$, $\epsilon < \pi$ such that in $(-\epsilon, \epsilon)$ $|\sin(x)| > A|x|$. This implies

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi}|K_n(x)|dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{\sin(nx)}{x^2}\right|dx &= \frac{n}{2\pi}\int_{-n\pi}^{n\pi}\left|\frac{\sin(y)}{y^2}\right|dy \\
&\ge \frac{n}{2\pi}\int_{-\pi}^{\pi}\left|\frac{\sin(y)}{y^2}\right|dy \\
&\ge \frac{n}{2\pi}\int_{-\epsilon}^{\epsilon}\frac{A|y|}{|y|^2}dy \\
&= \frac{nA}{2\pi}\int_{-\epsilon}^{\epsilon}\frac{1}{|y|}dy \\
&= \infty,
\end{aligned}$$

where we also made a substitution $y = nx$. Hence, condition (c) is not satisfied and the kernel $K_n(x) = \frac{\sin(nx)}{x^2}$ is not a family of good kernels.

(iii) For the last kernel we check the above conditions:

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi}K_n(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}(n+1)\Big(1-\frac{|x|}{\pi}\Big)^n dx &= \frac{n+1}{\pi}\int_0^{\pi}\Big(1-\frac{x}{\pi}\Big)^n dx \\
&= -\int_0^{\pi}\frac{d}{dx}\Big(1-\frac{x}{\pi}\Big)^{n+1}dx \\
&= 1, \quad \forall n
\end{aligned}$$

As $K_n = |K_n|$ this implies

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|K_n(x)|dx \le c, \quad \forall n \in \mathbf{N}.$$

Now, fix $\delta > 0, \delta < \pi$. Then

$$\begin{aligned}\int_{\delta\le|x|\le\pi} |K_n(x)|dx &= \int_{\delta\le|x|\le\pi} (n+1)\Big(1-\frac{|x|}{\pi}\Big)^n dx \\ &\le (n+1)\int_{\delta\le|x|\le\pi} \Big(1-\frac{\delta}{\pi}\Big)^n dx \\ &= (n+1)2(\pi-\delta)\underbrace{\Big(1-\frac{\delta}{\pi}\Big)^n}_{\in[0,1)} \to 0, \quad \text{as} \quad n\to\infty.\end{aligned}$$

Thus, all the conditions are satisfied and $K_n(x) = (n+1)(1-\frac{|x|}{\pi})^n$ is a family of good kernels. $\square$

4. Let f be a continuous and g an integrable 2π-periodic functions. Prove that then the convolution $f * g$ is continuous.

Proof. Let f to be continuous on $[-\pi,\pi]$ and $g \in L^1[-\pi,\pi]$. Since f is continuous on the compact set $[-\pi,\pi]$, it is in fact uniformly continuous on $[-\pi,\pi]$.
Since f is 2π-periodic and uniformly continuous on $[-\pi,\pi]$, it is in fact uniformly continuous on $\mathbf{R}$.
Fix a point $x_0 \in \mathbf{R}$ and $\epsilon > 0$. Since f is uniformly continuous, there exists $\delta > 0$ such that $|f(z) - f(z')| < \epsilon$, whenever $|z - z'| < \delta$. Now,

$$|f*g(x_0) - f*g(x)| = |g*f(x_0) - g*f(x)| \le \int_{-\pi}^{\pi} |g(y)||f(x_0-y) - f(x-y)|dy \le \epsilon\|g\|_{L^1[-\pi,\pi]}$$

whenever $|x - x_0| < \delta$. Thus, $f * g$ is continuous at x_0. $\square$

5. (i) Show that for every 2π-periodic function $f \in L^1[-\pi,\pi]$ we have

$$\widehat{f}(n) = \frac{1}{4\pi}\int_0^{2\pi} e^{-inx}(f(x) - f(x+\pi/n))dx.$$

(ii) If $f \in C_{\#}(-\pi,\pi)$ is Hölder continuous with exponent $\alpha \in (0,1]$ i.e., $|f(x) - f(y)| \le C_0|x-y|^\alpha$ with some constant C_0, show that

$$|\widehat{f}(n)| \le C|n|^{-\alpha}, \quad \text{for} \quad |n| \ge 1.$$

Proof. (i) Making a change of variable $x = t + \pi/n$ we find that

$$\begin{aligned}
\widehat{f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx} f(x)dx \\
&= \frac{1}{2\pi}\int_{-\pi-\pi/n}^{\pi-\pi/n} e^{-in(t+\pi/n)} f(t+\pi/n)dt \\
&= -\frac{1}{2\pi}\int_{-\pi-\pi/n}^{\pi-\pi/n} e^{-int} f(t+\pi/n)dt \\
&= -\frac{1}{2\pi}\int_{0}^{2\pi} e^{-int} f(t+\pi/n)dt
\end{aligned}$$

We were able to replace the interval $[-\pi - \pi/n, \pi - \pi/n]$ with $[0, 2\pi]$ in the last step by the 2π-periodicity of the function $e^{-int}f(t + \pi/n)$. Now we find the required formula by the calculation

$$\begin{aligned}
\widehat{f}(n) &= \frac{1}{2}\Big(\widehat{f}(n) + \widehat{f}(n)\Big) \\
&= \frac{1}{2}\Big[\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx} f(x)dx + \Big(-\frac{1}{2\pi}\int_{0}^{2\pi} e^{-int} f(t+\pi/n)dt\Big)\Big] \\
&= \frac{1}{4\pi}\int_{0}^{2\pi} e^{-inx}(f(x) - f(x+\pi/n))dx.
\end{aligned}$$

(ii) Suppose f is Hölder continuous with exponent α. Then

$$|f(x) - f(x+\pi/n)| \le C_0\Big|\frac{\pi}{n}\Big|^{\alpha} = C_0\pi^{\alpha}|n|^{-\alpha},$$

and hence by part (i)

$$|\widehat{f}(n)| \le \frac{1}{4\pi}\int_{0}^{2\pi} |f(x) - f(x+\pi/n)|dx \le C_0\frac{1}{4\pi}\int_{0}^{2\pi} \pi^{\alpha}|n|^{-\alpha}dx = C|n|^{-\alpha},$$

where $C = C_0\frac{1}{2}\pi^{\alpha}$ is a constant. □

6. From lectures we know that the Fourier coefficients $\widehat{f}(n)$ of a function $f \in C_{\#}^{(k)}$ decay to zero with the speed $O(|n|^{-k})$ as $n \to \infty$.
Show that a converse result holds in the following sense: If f is continuous on the interval $[-\pi, \pi]$ and for an integer $k \ge 2$ there is a constant $C = C_k$ for which

$$|\widehat{f}(n)| \le C_k(1 + |n|)^{-k}, \quad n \in \mathbf{Z}.$$

then $f \in C_{\#}^{k-2}(-\pi, \pi)$.

Proof. We will show that f is in C^1 and deduce the rest by induction. First of all, just from the case k = 2 we know that

$$|\widehat{f}(n)| \leq C_2(1+|n|)^{-2}$$

for some constant C_2. Thus the Fourier series of f converges absolutely, and hence by Theorem 2.8 the Fourier series converges uniformly to f on $[-\pi, \pi]$. We can hence write

$$f(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{inx}.$$

If we show that the series on the right hand side is continuously differentiable, then so is f. To do this we first define the function g by

$$g(x) = \sum_{n=-\infty}^{\infty} in\widehat{f}(n)e^{inx}.$$

Note that the series defining g is well-defined and converges uniformly to g since

$$|in\widehat{f}(n)| \leq C_3|n|(1+|n|)^{-3} \leq C_3(1+|n|)^{-2}$$

for another constant C_3. We want to justify saying that $f' = g$. For this, recall a result from Analysis course which says that if f_n is a sequence of functions converging to f at the point x and the sequence f_n' converges uniformly to another function g, then $f'(x) = g(x)$. Applying this result to the partial sums

$$f_N(x) = \sum_{n=-N}^{N} \widehat{f}(n)e^{inx} \quad , \quad f_N'(x) = \sum_{n=-N}^{N} in\widehat{f}(n)e^{inx}$$

shows that $f'(x) = g(x)$ for all x as wanted. Since g is a uniform limit of continuous functions, it is continuous. Thus f is in C^1. Moreover, for all $k \geq 3$ it holds that

$$|\widehat{g}(n)| = |in\widehat{f}(n)| \leq |n|C_k(1+|n|)^{-k} \leq C_k(1+|n|)^{-k+1}.$$

Hence $g = f'$ satisfies the same condition as the original function f. Thus by the same arguments as above g must also be in C^1, so f is in C^2. By induction we see that f has to be in C^{k-2} for all $k \geq 2$. □

7. Let $f : [0,1] \to \mathbf{R}$ be continuous. We define its 'moments' $M_f(n)$ for $n = 0, 1, 2, \ldots$ by the formula

$$M_f(n) := \int_0^1 x^n f(x)dx.$$

Show that the moments determine the function f uniquely, i.e if $g : [0,1] \to \mathbf{R}$ is another continuous function and $M_g(n) = M_f(n)$ for all indices $n \geq 0$, then $f(x) = g(x)$ for all $x \in [0,1]$.

Proof. Assume $x_0 \in (0,1)$. By considering $f - g$ it is enough to prove that if f is continuous at x_0 and $M_f(n) = 0$ for all indices $n \geq 0 \implies f(x_0) = 0$. Since f is real valued in order to get a contradiction we assume that $f(x_0) \neq 0$ and w.l.o.g $f(x_0) > 0$.
By continuity, choose $\delta > 0$ small enough: $f(x) \geq \frac{f(x_0)}{2}$ for $|x - x_0| \leq \delta$.
Define $p_0(x) := 2 - \epsilon - (x_0 - x)^2$, where $\epsilon > 0$ is so small that: $|p_0(x)| < \frac{5}{2}, \quad \forall x \in [0,1] \setminus [x_0 - \delta, x_0 + \delta]$.
Pick η so small that: $p_0(x) \geq 2 - 2\epsilon, \quad \forall \quad |x - x_0| \leq \eta < \delta$ (by continuity).
Set $p_k(x) =: p_0^k(x) = (2 - 2\epsilon)^k, \quad k \in \mathbf{N}$. For every polynomial $q(x)$ and for our f : $\int_0^1 f(x)q(x)dx = 0$. Especially, $\int_0^1 f(x)p_k(x)dx = 0$.
We have the following properties of p_k:

(i) $|p_k(x)| < \frac{5}{2}$, if $|x - x_0| \geq \delta$

(ii) $|p_k(x)| \geq (2 - 2\epsilon)^k$, if $|x - x_0| \leq \eta$

(iii) $p_k(x) \geq 0$, if $|x - x_0| \leq \delta$.

Hence:

$$
\begin{aligned}
0 = \int_0^1 f(x)p_k(x)dx &= \int_{[0,x_0-\delta]\cup[x_0+\delta,1]} f(x)p_k(x)dx + \int_{|x-x_0|\leq\eta} f(x)p_k(x)dx \\
&\quad + \int_{\eta\leq|x-x_0|\leq\delta} f(x)p_k(x)dx \\
&> 2\eta\frac{f(x_0)}{2}\big(2 - 2\epsilon\big)^k - \frac{5}{2}\int_0^1 |f(x)|dx \to \infty, \quad \text{as} \quad k \to \infty. \quad ※
\end{aligned}
$$

So the claim is proven. $\square$