FOURIER ANALYSIS I. (Spring 2020)

4. EXERCISES (Thursday 13.2 14-16 in room C123)

1. At which points $x \in [-\pi, \pi)$ does the Fourier series of the 2π-periodic function f converge (and to what value does it then converge - write down the identity you get!) when

(i) $f(x) := e^x$ for $x \in [-\pi, \pi)$;

(ii) $f(x) := x^2$ for $x \in [-\pi, \pi)$.

2. As an example of a concrete function with really rapidly converging Fourier series, prove the formula

$$e^{\cos x} \cos(\sin(x)) = \sum_{n=0}^{\infty} \frac{\cos(nx)}{n!}.$$

[Hint: Instead of computing Fourier series of the function it is easier to evaluate the series on the right hand side directly (recall the Taylor series of a very familiar function...).]

3. True or false: if $(x_n)_{n\geq 1}$ is equidistributed (mod 1) , and $\lim_{n\to\infty} \varepsilon_n \to 0$, then also the sequence $(x_n + \varepsilon_n)_{n\geq 1}$ is also equidistributed (mod 1).

4. Let $a > 0$ be a real number. Suppose the sequence $(x_n)_{n=1}^{\infty}$ is equidistributed (mod 1). Show that if $a \in \mathbf{Z} \setminus \{0\}$, then also the sequence $(ax_n)_{n=1}^{\infty}$ is equidistributed (mod 1).

Does the same result hold also for all $\alpha \notin \mathbf{Q}$?

5. Let $f \in C_{\#}(-\pi, \pi)$. Assume that f has another period $\beta > 0$: $f(\beta + x) = f(x)$ for all x. Show that f is constant if $\beta/2\pi$ is irrational.

6. Use the Dirichlet kernel $D_N(x) = \frac{\sin[(N+\frac{1}{2})\,x]}{\sin\frac{x}{2}}$ and the knowledge $\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(x)dx = 1$ to show that

$$\int_0^{\infty} \frac{\sin x}{x}\,dx := \lim_{M\to\infty} \int_0^M \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

[Hint: Show that the function $g(x) = \frac{1}{\sin\frac{x}{2}} - \frac{2}{x}$ is continuous on the interval $[-\pi, \pi]$, and use the Riemann-Lebesgue Lemma.]

7*[1] Suppose that f is the 2π–periodic function given by $f(x) = \operatorname{sgn}(x) = x/|x|$, when $0 < |x| \leq \pi$, and $f(0) = 0$. Recall the Fourier series of f (we did this already carefully at the lectures!), and show that if N is an odd integer,

$$S_N f(x) = \frac{4}{\pi} \sum_{k=0}^{(N-1)/2} \frac{\sin((2k+1)x)}{2k+1}.$$

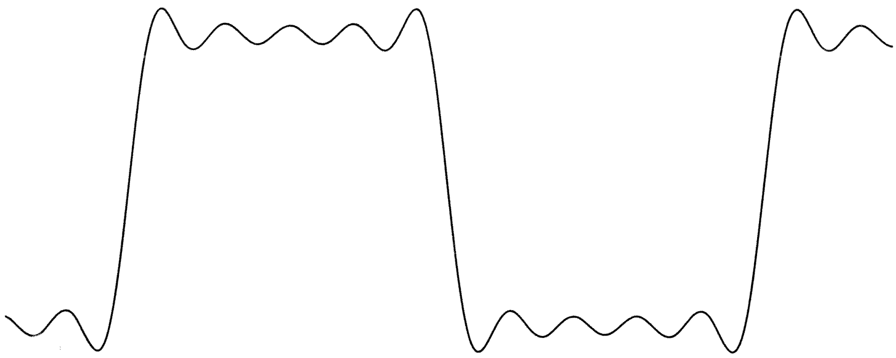
[1] These ∗-exercises are extras for afficinadoes, not required to get full points from exercises

Prove that

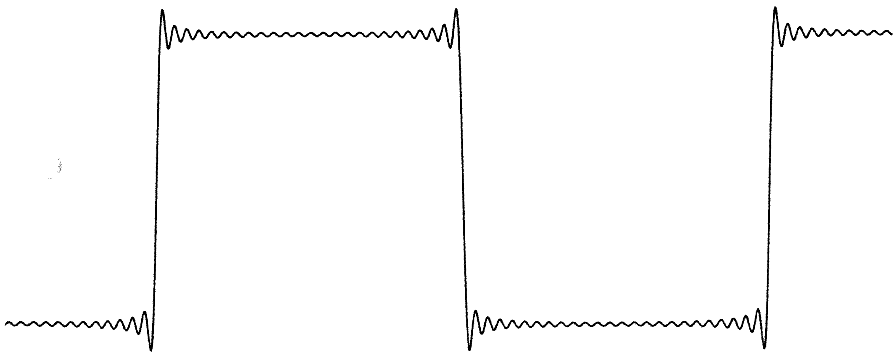
$$\lim_{N\to\infty} S_N f(\frac{\pi}{N}) = \frac{2}{\pi}\int_0^{\pi} \frac{\sin(x)}{x} dx =: G_0 > 1.$$

This proves the *Gibbs phenomenon* for the function f, c.f. the pictures on the next page.

[Hint: In " $>$ " make use Problem 6.]



$S_N f(x)$, $N = 5$



$S_N f(x)$, $N = 25$