Problem set 4, Advanced risk theory, 14.2.2020

Exercise 4.1. Give an example of a distribution of random variable ξ such that

$$\liminf_{x\to\infty} \frac{-\log \mathbb{P}(\xi > x)}{x} < \limsup_{x\to\infty} \frac{-\log \mathbb{P}(\xi > x)}{x}.$$

Exercise 4.2. Let F be a distribution function such that $F(0) = 0$ and $\bar{F}(y) = 1 - F(y) > 0$ for all $y \in \mathbb{R}$. Then F (or more precisely, the distribution determined by F) is called subexponential, if

$$\mathbb{P}(S_n > x) \sim n\mathbb{P}(\xi_1 > x), \quad x \to \infty,$$

for all $n = 1, 2, \ldots$, where $S_n = \xi_1 + \cdots + \xi_n$ and $\xi_1, \xi_2, \ldots$ are i.i.d. variables with common distribution function F.

Show that, when F is subexponential,

$$\lim_{x\to\infty} \mathbb{P}(\xi_1 > x \,|\, \xi_1 + \xi_2 > x) = \lim_{x\to\infty} \mathbb{P}(\xi_2 > x \,|\, \xi_1 + \xi_2 > x) = \frac{1}{2}.$$

How would you interpret this result?

Exercise 4.3 (Exercise 4.2 continues)**.** Let F be as in Exercise 4.2 and let $n \in \mathbb{N}$. Denote

$$M_n = \max(\xi_1, \ldots, \xi_n).$$

a) Show that

$$\mathbb{P}(M_n > x) \sim n\bar{F}(x), \quad x \to \infty.$$

b) Suppose, in addition, that F is subexponential. Show that

$$\mathbb{P}(M_n > x) \sim \mathbb{P}(S_n > x), \quad x \to \infty.$$

Exercise 4.4. Prove Lemma 3.3. of the lecture notes.

Exercise 4.5. Suppose $n = 2, 3, \ldots$ is given and let $S_n = \xi_1 + \cdots + \xi_n$ be the sum of i.i.d. non-negative random variables $\xi_1, \ldots, \xi_n$. Assume further that $\mathbb{I}(\xi_1) = \alpha$, where $\alpha \in (0, \infty)$.

Show that $\mathbb{I}(S_n) = \alpha$.