Algebra II

Exercise 4 (13.2.2020)

1. Suppose that G is a group.

a) Prove that every normal subgroup of G is the union of some conjugacy classes.

b) Prove that if $H \leq G$ and H is the union of some conjugacy classes, then H is a normal subgroup of G.

2. Prove the remaining part of Proposition 3.9.
Suppose that $\sigma \in S_n$, written as a product of disjoint cycles $\sigma = \rho_1 \cdots \rho_n$ (with the 1-cycles included). Suppose that among the cycles ρ_i there is at least one, which is of even length, or there are two cycles ρ_i, ρ_j, $i \neq j$, which are of same length. Prove that then there exists an odd permutation α, such that $\alpha\sigma\alpha^{-1} = \sigma$. (This means that $\alpha \in C_S \setminus C_A$, which gives us that $C_A \neq C_S$.)

3. Suppose that a group G acts on a set X and the elements $g, h \in G$ belong to the same conjugacy class. Prove that $|\mathrm{Fix}(g)| = |\mathrm{Fix}(h)|$.

4. Let p be a prime number. We say that a group G is a *p-group*, if $|G| = p^k$ for some $k \geq 1$. Prove that the center $Z(G)$ of every p-group G is non-trivial. [Hint: You can use the class equation

$$|G| = \sum_{x \in C} [G : C_G(x)],$$

where C is a set of representatives, one from each conjugacy class. What is $C_G(x)$, if $x \in Z(G)$?]

5. a) Let G be a group, with center $Z(G)$. Prove that if the quotient group $G/Z(G)$ is cyclic, then G is commutative.

b) Let p be a prime number. Prove that every group, whose order is p^2, is commutative. [Hint: Exercise 4]

6. Determine the conjugacy classes of the group S_4, their sizes, and all normal subgroups of S_4.