# Fourier Analysis I

Spring 2020

Homework 3

Exercise session: Thu 6 February, 14:15 - 16:00, Exactum C123; Stefanos Lappas, stefanos.lappas@helsinki.fi.

**1. (i)** Let $N \in \mathbb{N}$. Show that there exists a non-trivial function $f \in L^1[-\pi, \pi]$ such that $F_N * f(x) = 0$ for all $x$. **(ii)** Is there a non-trivial function $f \in L^1[-\pi, \pi]$ so that $F_N * f(x) = 0$ for all $x$ and for all $N \geq 0$ ?

*Proof.* (i) We can choose the function $f$ as $f(x) = e^{i(N+1)x}$. Now we have $\widehat{f}(n) = 0$ for any $n \neq N + 1$ and so

$$F_N * f(x) = \frac{1}{N+1} \sum_{k=0}^{N} f * D_k(x) = 0,$$

where $D_k(x) = \sum_{|n| \leq k} e^{inx}$ is the Dirichlet kernel.

(ii) There is no such function. As the Fejer kernels are a good sequence of kernels, we know that $\|f - F_N * f\|_{L^1(-\pi,\pi)}$ goes to zero as $N$ goes to infinity. Assuming that $F_N * f(x) = 0$ for all $x$ and for all $N$ gives us

$$\|f\|_{L^1(-\pi,\pi)} = \|f - F_N * f\|_{L^1(-\pi,\pi)} \to 0.$$

This means that $\|f\|_{L^1(-\pi,\pi)} = 0$, so $f(x) = 0$ for almost every $x$. □

**2. (i)** Show that if there exist the limit $A := \lim_{n\to\infty} a_n$, then also

$$\lim_{N\to\infty} \frac{a_0 + a_1 + \ldots + a_{n-1}}{N} = A$$

**(ii)** Use part (i) to verify that if the series $\sum_{n=0}^{\infty} b_n$ converges and has sum $S$, then it is also Cesaro summable, i.e. if $s_n := \sum_{k=0}^{n} b_n$, we have

$$S = \lim_{N\to\infty} \frac{s_0 + s_1 + \ldots + s_{n-1}}{N}.$$

Show by a counter example that the converse is not true.

*Proof.* (i) The idea is to split the average into the weighted average of finitely many "beginning" terms and the weighted average of infinitely many "terms" and note that the contribution of the beginning is vanishing whereas the tail approaches the limit.

Let $M < N$ be positive integers to be chosen later. We split the average by writing

$$\frac{a_0 + a_1 + \ldots + a_N}{N} = \Big(\frac{M+1}{N}\Big)\frac{a_0 + a_1 + \ldots + a_M}{M+1} + \Big(\frac{N-M}{N}\Big)\frac{a_{M+1} + \ldots + a_N}{N-M}.$$

Similarly,

$$A = \Big(\frac{M+1}{N}\Big)A + \Big(\frac{N-M}{N}\Big)A.$$

Thus, by substituting these splittings and applying triangle inequality,

$$\begin{aligned}\Big|\frac{a_0 + a_1 + \ldots + a_N}{N} - A\Big| &\le \Big(\frac{M+1}{N}\Big)\Big|\frac{(a_0 - A) + (a_1 - A) + \ldots + (a_M - A)}{M}\Big| \\ &\quad + \Big(\frac{N-M}{N}\Big)\Big|\frac{(a_{M+1} - A) + \ldots + (a_N - A)}{N-M}\Big| \\ &\le \Big(\frac{M+1}{N}\Big)\sup_{0\le n\le M}|a_n - A| + \Big(\frac{N-M}{N}\Big)\sup_{M+1\le n\le N}|a_n - A|.\end{aligned}$$

Now, we prove the claim of the exercise. Assume that $\lim_{n\to\infty} a_n = A$. Fix $\epsilon > 0$. Since $\lim_{n\to\infty} a_n = A$, we can choose $M$ so that $\sup_{M+1\le N\le M}|a_n - A| \le \epsilon$ for all $N \ge M$. Since $\lim_{N\to\infty}\frac{M+1}{N} = 0$, we can choose $N(M)$ so that $\frac{M+1}{N}\sup_{0\le n\le M}|a_n - A| \le \epsilon$ for all $N \ge M$. Altogether, for all $N > N_0 := \max\{M, N(M)\}$ we have

$$\Big|\frac{a_0 + a_1 + \ldots + a_N}{N} - A\Big| \le 2\epsilon.$$

Therefore, $\lim_{N\to\infty}\frac{a_0+a_1+\ldots+a_N}{N} = A$.

(ii) This follows from using the result of (i) to the sequence $s_n := \sum_{k=0}^n b_n$.

As a counterexample, consider the sequence $b_n = (-1)^n$. The partial sums are $s_N = \sum_{n=0}^N b_n = (1 + (-1)^N)/2$. As all the partial sums are alternatingly 1 or 0, the series is not summable. However, it is Cesaro summable: for even $N$, we have

$$\frac{s_0 + s_1 + \ldots + s_{n-1}}{N} = \frac{N/2}{N} = \frac{1}{2}$$

and for odd $N$

$$\frac{s_0 + s_1 + \ldots + s_{n-1}}{N} = \frac{(N+1)/2}{N} = \frac{1}{2} + \frac{1}{2N} \to \frac{1}{2}, \quad \text{as} \quad N \to \infty.$$

□

**3.** During the lectures it was shown that trigonometric polynomials are dense in $L^p(-\pi,\pi)$ if $1 \le p < \infty$. Is the same result true if $p = \infty$?

*Proof.* The result is not true in $L^\infty(-\pi,\pi)$. As the trigonometric polynomials are continuous, it is enough to show that the space of continuous functions $C(-\pi,\pi)$ is not dense in $L^\infty(-\pi,\pi)$. Choose $f \in L^\infty(-\pi,\pi)$ to be the sign function, in other words,

$$f(x) = \begin{cases} 1, & \text{if } \ 0 < x < \pi \\ 0, & \text{if } \ x = 0 \\ -1, & \text{if } \ -\pi < x < 0. \end{cases}$$

We will show that for any continuous function $g \in C(-\pi,\pi)$, we have $\|f - g\|_{L^\infty(-\pi,\pi)} \ge 1/2$ : this would prove the claim.
Let $g$ be a continuous function $g \in C(-\pi,\pi)$. As $g$ is continuous at 0, we can find $0 < \delta < \pi$ such that

$$|g(x) - g(0)| < 1/2 \quad \text{whenever} \quad |x| < \delta.$$

First assume $g(0) \ge 0$. Now for any $-\delta < x < 0$ we can apply triangle inequality to get

$$|g(x) - f(x)| = |g(x) - g(0) + g(0) + 1| \ge |g(0) + 1| - |g(x) - g(0)| \ge |g(0) + 1| - 1/2 \ge 1/2.$$

As the set $(-\delta, 0)$ has measure $\delta > 0$, we get $\|f - g\|_{L^\infty(-\pi,\pi)} \ge 1/2$. Similarly, if $g(0) < 0$, for $0 < x < \delta$ we have

$$|g(x) - f(x)| = |g(x) - g(0) + g(0) - 1| \ge |g(0) - 1| - |g(x) - g(0)| \ge |g(0) - 1| - 1/2 \ge 1/2$$

and the set $(0, \delta)$ has measure $\delta > 0$. □

**4.** Define $f(x) = 0$ for $x \in [-\pi, 0]$, $f(x) = \pi - x$ for $x \in [0, \pi)$, and extend $f$ to $2\pi$-perodic function. Compute the Fourier series of $f$. In which points does the Fourier series of the function $f(x)$ converge and to what value?

*Proof.* We compute the Fourier coefficients. First, if $n = 0$, we have

$$\widehat{f}(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi} \int_0^{\pi} (\pi - x)dx = \frac{\pi}{2} - \frac{1}{2\pi} \int_0^{\pi} xdx = \frac{\pi}{4}.$$

For $n \neq 0$, we can use integration by parts:

$$\begin{aligned}
\widehat{f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \\
&= \frac{1}{2\pi}\int_{0}^{\pi} (\pi - x)e^{-inx}dx \\
&= \frac{1}{2}\int_{0}^{\pi} e^{-inx}dx - \frac{1}{2\pi}\int_{0}^{\pi} xe^{-inx}dx \\
&= \frac{1}{-2in}(e^{-in\pi} - 1) - \frac{1}{2\pi}\Big(\pi\frac{1}{-in}e^{-in\pi} - \int_{0}^{\pi}\frac{1}{-in}e^{-inx}dx\Big) \\
&= \frac{i}{2n}((-1)^n - 1) - \frac{i}{2n}(-1)^n + \frac{1}{2\pi}\int_{0}^{\pi}\frac{1}{-in}e^{-inx}dx \\
&= \frac{-i}{2n} + \frac{-1}{2\pi n^2}(e^{-in\pi} - 1) \\
&= \frac{-i}{2n} + \frac{1 - (-1)^n}{2\pi n^2}.
\end{aligned}$$

The simplest way to see that the Fourier series converges everywhere is by using Dini's criterion. We define function $g : [-\pi, \pi] \to \mathbb{C}$ by setting $g(0) = \pi/2$ and $g(x) = f(x)$ for any $x \neq 0$. As $g$ and $f$ coincide almost everywhere, they have the same Fourier coefficients. After extending $g$ to be $2\pi$-periodic, we will show that for any $x_0 \in [-\pi, \pi]$, Dini's criterion holds at $x_0$ so the Fourier series converges to $g(x_0)$.
For $-\pi < x_0 < 0$, write $\delta = \min(x_0 + \pi, -x_0)$. We have $g(x_0 + t) = 0$ whenever $|t| < \delta$, so we get

$$\int_0^{\pi}\Big|\frac{g(x_0+t)+g(x_0-t)}{2} - g(x_0)\Big|\frac{dt}{t} \leq \frac{1}{\delta}\int_{\delta}^{\pi}\Big|\frac{g(x_0+t)+g(x_0-t)}{2} - g(x_0)\Big|dt < \infty.$$

For $0 < x_0 < \pi$ write $\delta = \min(\pi - x_0, x_0)$. We have $g(x_0 + t) = \pi - (x_0 + t)$ whenever $|t| < \delta$, so we get

$$\begin{aligned}
\int_0^{\pi}\Big|\frac{g(x_0+t)+g(x_0-t)}{2} - g(x_0)\Big|\frac{dt}{t} &\leq \int_0^{\delta}\Big|\frac{\pi-(x_0+t)+\pi-(x_0-t)}{2} - (\pi - x_0)\Big|\frac{dt}{t} \\
&\quad + \frac{1}{\delta}\int_{\delta}^{\pi}\Big|\frac{g(x_0+t)+g(x_0-t)}{2} - g(x_0)\Big|dt < \infty.
\end{aligned}$$

For $x_0 = 0$ we have

$$\int_0^{\pi}\Big|\frac{g(t)+g(-t)}{2} - g(0)\Big|\frac{dt}{t} = \int_0^{\pi}\Big|\frac{\pi - t}{2} - \frac{\pi}{2}\Big|\frac{dt}{t} = \int_0^{\pi}\frac{1}{2}dt = \frac{\pi}{2} < \infty.$$

For $x_0 = \pi$ we have

$$\int_0^\pi \Big|\frac{g(\pi+t)+g(\pi-t)}{2} - g(\pi)\Big|\frac{dt}{t} = \int_0^\pi \Big|\frac{\pi-(\pi-t)}{2}\Big|\frac{dt}{t} = \int_0^\pi \frac{1}{2}dt = \frac{\pi}{2} < \infty.$$

We have obtained that the Fourier series of f converges everywhere on the interval $[-\pi, \pi]$. For $x \neq 0$ it converges to $f(x)$ and it converges to $\frac{\pi}{2}$ at 0. □

**5.** Provide more details to the proof of the result sketched at the lectures: Corollary 4.8; that is, show that if a $2\pi$-periodic function $f(x)$ is piecewise [1] $C^1$ , then its Fourier series converges at every point, and

$$\lim_{N\to\infty} S_N f(x) = \lim_{t\to 0} \frac{f(x+t)+f(x-t)}{2}, \qquad x \in [-\pi, \pi].$$

*Proof.* Let $f$ be piecewise $C^1$, so there exists a partition $-\pi = x_0 < x_1 < \cdots < x_n = \pi$ such that the restrictions $f|(x_{j-1}, x_j)$ are $C^1$ and the one-sided limits $\lim\limits_{x\to x_{j-1}^+} f'(x)$ and $\lim\limits_{x\to x_j^-} f'(x)$ exist for all $j = 1, \ldots, n$. We lose no generality if we only show the convergence at the point $x_0 = 0$, since we can always make a substitution $F(x) = f(x + x_0)$ as in the proof of the Dini criterion. We hence want to show

$$\lim_{N\to\infty} S_N f(0) = \frac{f(0+)+f(0-)}{2}, \tag{1}$$

where $f(0+)$ and $f(0-)$ denote the right and left limits of the function $f$ at the point $x = 0$. For a piecewise $C^1$-function these limits always exist but may disagree. We now apply Lemma 4.2 of the lecture notes. By the lemma, to show that (1) holds it is enough to check that

$$\int_0^\pi \Big|\frac{f(x)+f(-x)}{2} - \frac{f(0+)+f(0-)}{2}\Big|\frac{dx}{x} < \infty.$$

It will be enough to show that the functions

$$g_+(x) = \frac{f(x)-f(0+)}{x} \quad \text{and} \quad g_-(x) = \frac{f(-x)-f(0-)}{x}$$

are bounded on the interval $(0, \pi)$, since

$$\int_0^\pi \Big|\frac{f(x)+f(-x)}{2} - \frac{f(0+)+f(0-)}{2}\Big|\frac{dx}{x} = \int_0^\pi \frac{|g_+(x)+g_-(x)|}{2}dx.$$

[1] i.e., apart from finitely many points $\{x_1, x_2, \ldots, x_n\} \subset [-\pi, \pi)$ the derivative $f'(x)$ exists and is continuous at $x$, and the derivative (and hence the function itself) has left and right limits at $x_j$ for all $j = 1, \ldots n$.

Let us take $\epsilon > 0$ sufficiently small so that $f$ is $C^1$ on the interval $(0, \epsilon]$. Then for $\epsilon \leq x \leq \pi$ we have that

$$|g_+(x)| \leq \frac{|f(x) - f(0+)|}{\epsilon},$$

which is bounded since f is bounded. When $0 < x < \epsilon$, we use the mean value theorem to find, for each $x$, a point $\xi_x \in (0, x)$ such that

$$g_+(x) = \frac{f(x) - f(0+)}{x} = f'(\xi_x).$$

Since $f$ is piecewise $C^1$, by definition the derivative $f'(x)$ is continuous on the closed interval $[0, \epsilon]$ and hence bounded. This shows that $g_+(x)$ is bounded, and by the same arguments so is $g_-(x)$. Hence the result is proven. □

**6.** Use the results of lectures so far to prove rigorously that every function $f : [0, \pi] \to \mathbb{C}$ that is Lipschitz-continuous (i.e. $|f(x) - f(y)| \leq C|x - y|$ for some $C < \infty$) and satisfies $f(0) = f(\pi) = 0$ can at each point $x \in [0, \pi]$ be expressed as a convergent sine series

$$f(x) = \sum_{k=1}^{\infty} c_k \sin(kx).$$

Find an expression for the coefficients of $c_k$.

*Proof.* Let us continue $f$ on the interval $[-\pi, \pi]$ by setting $f(x) = -f(-x)$ when $-\pi \leq x \leq 0$, so $f$ is an odd function. Now by exercise sheet 1 we can represent the Fourier series of $f$ as a sine series

$$f(x) = \sum_{n=1}^{\infty} 2i\widehat{f}(n) \sin(nx).$$

The coefficients of the sine series are given by

$$\begin{aligned} c_n = 2i\widehat{f}(n) &= \frac{i}{\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \\ &= \frac{i}{\pi} \int_{-\pi}^{0} f(x) e^{-inx} dx + \frac{i}{\pi} \int_{0}^{\pi} f(x) e^{-inx} dx \\ &= \frac{i}{\pi} \int_{0}^{\pi} f(x)(e^{-inx} - e^{inx}) dx = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin(nx) dx. \end{aligned}$$

We now consider the convergence of the Fourier series. We apply Dini's criterion. For $0 < x_0 < \pi$, writing $\delta = \min(x_0, \pi - x_0)$ we have

$$\begin{aligned}
\int_0^\pi \left|\frac{f(x_0+t)+f(x_0-t)}{2} - f(x_0)\right|\frac{dt}{t} &= \int_0^\delta \left|\frac{f(x_0+t)+f(x_0-t)}{2} - f(x_0)\right|\frac{dt}{t} \\
&+ \int_\delta^\pi \left|\frac{f(x_0+t)+f(x_0-t)}{2} - f(x_0)\right|\frac{dt}{t} \\
&\leq \int_0^\delta \frac{|f(x_0+t)-f(x_0)|+|f(x_0-t)-f(x_0)|}{2}\frac{dt}{t} \\
&+ \frac{1}{\delta}\int_\delta^\pi \frac{|f(x_0+t)|+|f(x_0-t)|}{2} + |f(x_0)|dt \\
&\leq C\delta + \frac{1}{\delta}\Big(\pi|f(x_0)| + 2\int_0^\pi |f(t)|dt\Big) < \infty,
\end{aligned}$$

where in the last inequality we used the fact that $f$ is Lipschitz-continuous and odd on the interval $[-\pi, 0]$. This proves that the Fourier series converges to $f(x_0)$. We have shown that $f$ can be represented as a convergent sine series

$$f(x) = \sum_{n=1}^\infty c_n \sin(nx)$$

with coefficients

$$c_n = \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)dx.$$

□

**7.** Can you find an example of a continuous function $f \in C_\sharp(-\pi,\pi)$ such that $S_n f(0) \to \infty$ as $n \to \infty$. An easier question: can you find an example of an integrable function $f \in L^1(-\pi,\pi)$ such that $S_n f(0) \to \infty$ as $n \to \infty$ ?

*Proof.* There is no such function. If $f \in C_\sharp(-\pi,\pi)$ then

$$\lim_{n\to\infty} S_n f(x) = \lim_{n\to\infty} \sigma_n f(x) = f(x)$$

for every $x \in (-\pi,\pi)$. This follows by exercise 2 (ii) and the fact that $\lim_{n\to\infty}\sigma_n f(x) = f(x)$ for every $x \in (-\pi,\pi)$.
For the second question we consider the function $f(x) = \sum_{n=1}^\infty \frac{1}{n^2}F_{n^3}(x)$, where $F_n$ is the Fejer kernel. Then $f \in L^1(-\pi,\pi)$, since $\|F_n\|_{L^1(-\pi,\pi)} = 1$ for all $n$. Using the fact that

$(S_nF_n)(0) = F_n(0) = n$ and the Fourier coefficients of the Fejer kernel are positive we have

$$(S_{k^3}f)(0) \geq S_{k^3}\left(\frac{1}{k^2}F_{k^3}\right)(0) = k \to \infty.$$

$\square$