Problem set 5, Advanced risk theory, 21.2.2020

Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers and let $\underline{a}$ and $\overline{a}$ be constants such that $0 < \underline{a} \leq \overline{a} < \infty$. Assume further that the members of the sequence $(a_n)_{n=1}^{\infty}$ satisfy $\underline{a} \leq a_n \leq \overline{a}$ for all $n \in \mathbb{N}$.

Consider the following slightly modified setting of Theorem 3.1. Suppose ξ_n, the loss of year n, is of the form $\xi_n = a_n \eta_n$, where $\eta, \eta_1, \eta_2, \ldots$ is an i.i.d. sequence of random variables. It is assumed that $\mathbb{E}(\eta) \in (-\infty, 0)$ and that

$$\lim_{x\to\infty} \frac{\log \mathbb{P}(\eta > x)}{\log x} = -\alpha,$$

where $\alpha \in (1, \infty)$.

Exercise 5.1. Let $\delta \in (0,1)$ be given and denote $h = h_n = n^{-1+\delta/2}$. For $j \in \mathbb{N}$, set

$$\xi_j' = \xi_j \mathbb{1}\left(\xi_j \leq n^{1-\delta}\right).$$

Show that it is possible to determine constants $\varepsilon' > 0$ and n_0, that do not depend on j, such that

$$\mathbb{E}\left(e^{h\xi_j'}\right) \leq e^{-h\varepsilon'}$$

holds for all $j \in \mathbb{N}$ and all $n \geq n_0$.

Exercise 5.2 (Exercise 5.1 continues)**.** Show that

$$\lim_{n\to\infty} \frac{\log \mathbb{P}(Y_n > 0, M_n \leq n^{1-\delta})}{\log n} = -\infty.$$

Exercise 5.3 (Exercise 5.2 continues)**.** Suppose $b > 0$ is given. Show that

$$\lim_{U_0\to\infty} \frac{\log \mathbb{P}(T \leq bU_0, M_{\lfloor bU_0 \rfloor} \leq U_0^{1-\delta})}{\log U_0} = -\infty.$$

Exercise 5.4 (Exercise 5.3 continues)**.** Show that

$$\limsup_{U_0\to\infty} \frac{\log \mathbb{P}(T \leq bU_0)}{\log U_0} \leq 1 - \alpha.$$

Exercise 5.5 (Exercise 5.4 continues)**.** Assume further that $\mathbb{P}(\eta > -c) = 1$ for some $c > 0$. Show that

$$\lim_{U_0\to\infty} \frac{\log \mathbb{P}(T \leq bU_0)}{\log U_0} = 1 - \alpha.$$