Algebra II. Exercise 4.
Solutions.

1. a) Suppose N is a normal subgroup of G. To prove the claim, it is sufficient to notice that if N contains an element n, then it contains all the elements in the conjugacy class of n. This is true, since by normality we have $gng^{-1} \in N$ for all $n \in N, g \in G$.
b) Let $h \in H$. From the assumption it follows that H contains the whole conjugacy class of h, but this means that $ghg^{-1} \in H$ for all $g \in G$. This holds for all $h \in H$, which proves that H is normal.

2. Claim 1: If the length of some cycle ρ_i is even, then there exists $\alpha \in C_S$, such that $\mathrm{sgn}(\alpha) = -1$ (that is, $\alpha \notin C_A$).
Proof: Choose $\alpha = \rho_i$, then we have $\alpha\rho_i\alpha^{-1} = \alpha = \rho_i$ (and also $\alpha\sigma\alpha^{-1} = \sigma$). Now α is a cycle, whose length is even, and thus $\mathrm{sgn}(\alpha) = -1$.
Claim 2: If in the permutation σ there exist two cycles $\rho_i, \rho_j,\ i \neq j$, having the same length, then there exists $\alpha \in C_S$, such that $\mathrm{sgn}(\alpha) = -1$.
Proof: If the length of the cycles ρ_i, ρ_j is even, then we can find α as in Claim 1. Suppose then, that the length of the cycles ρ_i, ρ_j is odd; $\rho_i = (a_1\ a_2\ \cdots\ a_k)$, $\rho_j = (b_1\ b_2\ \cdots\ b_k)$, k odd. If we denote $\alpha = (a_1\ b_1) \cdots (a_k\ b_k)$, then $\mathrm{sgn}(\alpha) = (-1)^k = -1$, because k is odd. Furthermore $\alpha\sigma\alpha^{-1} = \sigma$ by Proposition 3.6.

3. Suppose that the elements $g, h \in G$ belong to the same conjugacy class, that is, there exists $a \in G$, for which $aga^{-1} = h$. Define the function

$$f\colon \mathrm{Fix}(g) \to \mathrm{Fix}(h)$$

$$x \mapsto ax.$$

We prove first that $ax \in \mathrm{Fix}(h)$, if $x \in \mathrm{Fix}(g)$:

$$h(ax) = (aga^{-1})(ax) = (aga^{-1}a)x = (ag)(x) = a(gx) = ax,$$

which proves the claim (in the last step we used the assumption $x \in \mathrm{Fix}(g)$). Similarly we can define

$$f'\colon \mathrm{Fix}(h) \to \mathrm{Fix}(g)$$

$$x \mapsto a^{-1}x$$

and check that $a^{-1}x \in \mathrm{Fix}(g)$, if $x \in \mathrm{Fix}(h)$. Clearly

$$f(f'(x)) = f(a^{-1}x) = a(a^{-1}x) = x \text{ ja } f'(f(x)) = ... = x,$$

hence f and f' are inverse functions of each other. Thus they are bijective, and the claim $|\mathrm{Fix}(g)| = |\mathrm{Fix}(h)|$ follows from this.

4. Observe first that

$$x \in Z(G) \Leftrightarrow C_G(x) = G \Leftrightarrow [G : C_G(x)] = 1.$$

From Lagrange's theorem it follows that $[G : C_G(x)] | p^m$, that is, $[G : C_G(x)] \in \{1, p, ..., p^m\}$. If $Z(G)$ were trivial, then there would exist only one x, for which $[G : C_G(x)] = 1$, namely $x = e$. From the class equation we would then get

$$p^m = |G| = 1 + p^{k_1} + p^{k_2} + \cdots p^{k_r},$$

where $k_1, ..., k_r \geq 1$. This is a contradiction, since the left hand side of the equation is p^m, which is divisible by p, but the right hand side is not.
Thus $Z(G)$ cannot be trivial.

5. a) Suppose that $G/Z(G)$ is cyclic and the element $gZ(G)$ generates it. Let $x, y \in G$. We prove that $xy = yx$. Now $x \in xZ(G)$ and $y \in yZ(G)$, so it follows from cyclicity, that $xZ(G) = g^m Z(G)$ and $yZ(G) = g^n Z(G)$ for some $m, n \in \mathbb{Z}$ and thus $x = g^m z_1$, $y = g^n z_2$ for some $z_1, z_2 \in Z(G)$. Now

$$xy = g^m z_1 \cdot g^n z_2 = g^m g^n z_1 z_2 = g^n g^m z_2 z_1$$

$$= g^n z_2 \cdot g^m z_1 = yx.$$

We used the fact that $z_1, z_2 \in Z(G)$, that is, they commute with every element of G.
b) Suppose that $|G| = p^2$. From Exercise 4 it follows that $Z(G) \neq \{e\}$, hence $|Z(G)| = p$ or p^2. If $|Z(G)| = p^2$, then $G = Z(G)$ is commutative. If $|Z(G)| = p$, then $|G/Z(G)| = p$. Because a group having prime order is cyclic, we have that $G/Z(G)$ is cyclic. Thus by item a) G is commutative.

6. The conjugacy classes are determined by the cycle types, and they are the following:

$C_1 = \{(1)\}$
$C_2 = \{(1\ 2), (1\ 3), (1\ 4), (2\ 3), (2\ 4), (3\ 4)\}$
$C_3 = \{(1\ 2\ 3), (1\ 2\ 4), (1\ 3\ 2), (1\ 3\ 4), (1\ 4\ 2), (1\ 4\ 3), (2\ 3\ 4), (2\ 4\ 3)\}$
$C_4 = \{(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$
$C_5 = \{(1\ 2\ 3\ 4), (1\ 2\ 4\ 3), (1\ 3\ 2\ 4), (1\ 3\ 4\ 2), (1\ 4\ 2\ 3), (1\ 4\ 3\ 2)\}$

The sizes of the conjugacy classes are: $|C_1| = 1,\ |C_2| = 6,\ |C_3| = 8,\ |C_4| = 3$ and $|C_5| = 6$.

Next we determine all the normal subgroups H.
We know that normal subgroups are unions of conjugacy classes. Also we know that the order of a subgroup is a factor of $|S_4| = 4! = 24$, and thus the possible orders for H are $1, 2, 3, 4, 6, 8, 12$ and 24. Every subgroup contains the class C_1.

If the order of H is odd, it is necessarily 1 or 3. The order 3 cannot be obtained as a union of conjugacy classes, so the only possibility is the trivial subgroup $\{(1)\}$.

If the order of H is even, we notice that (since always $(1) \in H$) we must have $C_4 \subset H$ (in order to get an even order). The following are all possibilities:
1) The set $C_1 \cup C_4$ is a subgroup (this is easy to verify); it is a normal subgroup, since it is a union of conjugacy classes.
2) The set $C_1 \cup C_3 \cup C_4 = A_4$ is a normal subgroup.
3) In the set $C_1 \cup C_2 \cup C_4$ there are 10 elements, so it cannot be a subgroup.
4) In the set $C_1 \cup C_4 \cup C_5$ there are 10 elements, so it cannot be a subgroup.
5) If we take the union of four conjugacy classes, then we have more than 12 elements, so it cannot be a subgroup.
6) Taking the union of all conjugacy classes we obtain the whole S_4.

Thus the normal subgroups are:

$$\{(1)\},\ \{(1), (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\},\ A_4 \text{ and } S_4.$$