Algebra II

Exercise 5 (20.2.2020)

1. Suppose that G is a group and $H \leq G$. Denote by $\mathcal{A} = \{gH \mid g \in G\}$ and $\mathcal{B} = \{Hg' \mid g' \in G\}$ the collections of the left and right cosets of H, respectively. Prove that we have a bijective map $\varphi\colon \mathcal{A} \to \mathcal{B}$ defined by $\varphi(gH) = Hg^{-1}$.

2. a) Give an example of a group G and subgroups $H, K \leq G$, such that $HK = \{hk \mid h \in H, k \in K\}$ is not a subgroup of G.

b) Give an example of a group G and subgroups $H, K \leq G$, such that $HK = G$ and $H \cap K = \{e_G\}$, but G is not isomorphic to the direct product $H \times K$.

3. Let G be a group. Prove that the following conditions are equivalent:
(i) $G \neq \{1\}$ and the only subgroups of G are G and $\{1\}$.
(ii) $G \cong \mathbb{Z}_p$, where p is a prime number.
(iii) $|G|$ is a prime number.
(iv) G is simple and commutative.

4. Prove the Cauchy Theorem: If a prime number p divides the order of a group G, then there exists $g \in G$, which has order p.

5. a) Suppose that $m, n \in \{2, 3, ...\}$ and $\gcd(m, n) = 1$. Prove that

$$\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n.$$

[Hint: Consider the element $([1]_m, [1]_n) \in \mathbb{Z}_m \times \mathbb{Z}_n$.]

b) Suppose that $n \in \{2, 3, ...\}$. Prove that the groups $\mathbb{Z}_n \times \mathbb{Z}_n$ and $\mathbb{Z}_{n^2}$ are not isomorphic. [Hint: Consider subgroups of order n.]

6. Let p be a prime number. Prove that every group G of order p^2 is isomorphic either to the group $\mathbb{Z}_{p^2}$ or to the group $\mathbb{Z}_p \times \mathbb{Z}_p$. [Hint: By Cauchy's Theorem G has a subgroup H of order p. Choose $x \in G \setminus H$ and investigate the possible orders of the element x.]