FOURIER ANALYSIS I. (Spring 2020)

5. EXERCISES (Thursday 20.2 14-16 in room C123)

1. Denote $f(x) = ax + 1$ and $g(x) = x^2$. Determine coefficient a so that the L^2-distance $\|f - g\|_{L^2(-\pi,\pi)}$is minimized. Try to make a geometric interpretation (in $L^2(-\pi, \pi)$) for the solution.

[Hint: this time using Fourier series is perhaps not the easiest way...]

2 Let $f \in C^1_\#$ and $\int_{-\pi}^{\pi} f(x)dx = 0$. Prove Poincare type inequality.

$$\int_{-\pi}^{\pi} |f(x)|^2 dx \ \leq \ \int_{-\pi}^{\pi} |f'(x)|^2 dx.$$

For which functions do you have equality here?

[Hint: recall the Fourier series of f'.]

3. Let $f \in L^2(-\pi, \pi)$. Find the trigonometric polynomial $p(x) := \sum_{n=-N}^{N} c_n e^{inx}$ which is closest to f in L^2-norm, i.e. find the coefficients c_n that minimise the quantity

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \Big| f(x) - \sum_{n=-N}^{N} c_n e^{inx} \Big|^2 dx$$

4. Suppose $f \in C^1_\#(-\pi, \pi)$. Show that the Fourier series of f converges absolutely, i.e. we have $\sum |\widehat{f}(n)| < \infty$.

[Hint: Determine $\widehat{f}(n)$ in terms of $\widehat{f'}(n)$, and recall that the Cauchy-Schwarz inequality holds for the inner product $(a, b)_{\ell^2} = \sum_{k=-\infty}^{\infty} a_k \overline{b_k}$ in the space ℓ^2, see Lecture notes p. 58 (the C-S holds in every inner product space).]

5. Solve – to find a formal solution formula is enough – by using Fourier series the following PDEs. Here $x \in [0, 2\pi)$, $t \geq 0$ and $x \to u(x, t)$ is assumed to be 2π-periodic, and the solution satisfies the initial value $u(x, 0) = f(x)$, where $f \in L^2(-\pi, \pi)$ (of course you may assume that f is 2π-periodic). Instead of sine series, since we are in the 2π-periodic case, use just standard Fourier series to make the 'Ansatz' $u(x, t) = \sum_{n \in \mathbf{Z}} A_n(t) e^{inx}$.

(i) $\frac{d}{dt} u(x, t) = -\big(\frac{d}{dx}\big)^4 u(x, t)$.

(ii) $\frac{d}{dt} u(x, t) = \frac{d}{dx} u(x, t)$.

Try to guess (no rigorous rigorous reasoning needed here) which one of the equations in the previous equation has the property that the solution is always smooth in the variable x for $t > 0$.

6. Compute the Fourier series of $f(x) = x^2$, $x \in (-\pi, \pi)$ and compute the L^2-norm of f in two ways: first by direct computation and then using the Fourier-coefficients. Use this to compute the $\sum_{n=1}^{\infty} n^{-4}$.

7*[1] Try to analyse rigorously the behaviour of the solutions u in problem 5. Especially, could you write a simpler solution formula (than using Fourier series) for the solution in part (ii) ?

[1]These ∗-exercises are extras for afficinadoes, not required to get full points from exercises