FOURIER ANALYSIS I, Spring 2020

Lecturer: Eero Saksman

Timetable of the lectures:

The 3 weekly lectures are on Mondays (B322), Tuesdays (C322), and Thursdays (C122) at 10-12. See for the weekly exercise class (Thursdays 14-16 C123) below. During some weeks there are less lectures. Here is the preliminary schedule (– means no lecture on that day) **PLEASE, FOLLOW THIS PAGE FOR CHANGES !!**:

Week	Monday	Tuesday	Thursday
3	−−	−−	16.1. = **First Lecture**!
4	20.1.	−−	23.1.
5	27.1.	28.1.	30.1.
6	3.2.	4.2.	−−
7	10.2.	11.2.	13.2.
8	17.2.	18.2.	20.2.
9	−− .	−−	−−

Exercise classes:

Exercise classes are given by Stefanos Lappas (`stefanos.lappas@helsinki.fi`), They are held on **Thursdays 14-16** in room C123. By exercises alone one can gather extra points for the exam (8/24 extra points at maximum). The exercises will appear at the webpage in section 'Materials'.

Blog

`Abreviations` [StSh] refers to E.M. Stein & R. Shakarchi: *Fourier Analysis, An Introduction*, Princeton University Press, 2003. [FiLN] refer to the Finnish lecture notes (we refer to these rarely, ask for a Finnish speaking student for help in translation if needed!).

`Thursday 16.1` Introduction: historical background of Fourier series. Computation of Fourier coefficients of trigonometric polynomials – the orthogonality relation of exponentials e^{inx}, $n \in \mathbb{Z}$. Definition of a Fourier series of a general integrable function. Interpretation as 2π-periodic functions on $\mathbb{R}$ (reduction of arbitrary period $L > 0$ to this case). For $k \geq 0$, spaces $C^k_{\#}(-\pi, \pi)$ - i.e. of functions whose 2π-periodic extension belongs to $C^k(\mathbb{R})$, i.e. it is k times continuously differentiable on $\mathbb{R}$. $\big($ [StSh] Chapter 1 (not in detail, but as a historic overview), pp. 29-26.$\big)$

`Monday 20.1` Fourier-series determines the function at any point of continuity of the function. The Fourier series of a continuous function $f \in C_\sharp(-\pi,\pi)$ with absolutely converging Fourier series (i.e. $\sum_{n=-\infty}^{\infty} |\widehat{f}(n)| < \infty$) converges to $f(x)$ for every x. $\big($ [StSh] pp. 39–42.$\big)$

`Thursday 23.1` Fourier-series of the derivative. The smoother the function, the faster the decay of the Fourier coefficients. The Fourier series of a twice continuously differentiable function f converges to $f(x)$ at each x. Convolutions of 2π-periodic functions. L^p-spaces. Dirichlet kernel. Good kernels. Properties of convolution. $\big($ [StSh] pp. 42–48.$\big)$

`Monday 27.1` Approximations by good kernels for continuous or L^p-functions. Formula fro the Dirichlet kernel, and its 'badness'. Cesaro summation. Fejer-kernel. $\big($ [StSh] pp. 48–52, [FiLN] pp. 25–26$\big)$

`Tuesday 28.1` Fejer-kernels are a good family. Consequences: Fejer-partials sums converge uniformly if f is continuous, and in L^p $(1 \leq p < \infty)$ if $f \in L^p$. Density of trigonometric polynomials in L^p. Riemann-Lebesgue lemma. Convergence of a the Fourier series of a simple jump function at the jump point. $\big($ [StSh] pp. 51-54, [FiLN] pp. 30-32.$\big)$

`Thursday 30.1` Criterion for convergence in a point ([FiLN, Lemma 4.2]), convergence of Fourier series of Dini-continuous functions or Hölder continuous functions. Convergence at points of differentiability. Convergence for piecewise continuously differentiable functions. Locality: convergence at a point depends only on the behaviour of the function near the point. $\big($ [StSh] pp. 81–83, [FiLN] pp. 32-37.$\big)$

`Monday 3.2` Wrap up of convergence results proven in the lectures. Example of a continuous functions whose Fourier series does not converge at a point. Discussion of known results (Kolmogorov's a.e. counterexample for L^1, Carleson-Hunt a.e. positive result for L^p-functions). $\big($ [StSh] pp. 83-87.$\big)$

`Tuesday 4.2` Equidistribution of sequences (mod 1). Equivalence with numerical integration of continuous 1-periodic functions. Weyl's criterion. Equidistribution of the sequence $(n\alpha)_{n\geq 1}$. $\big($ [StSh] pp. 105–113.$\big)$

`Monday 10.2` Hilbert spaces. L^2-theory of Fourier series: Parseval equality, density of exponentials, they are an ON-basis, partial sum gives the best approximation among trig. polynom. of the same degree .$\big($ [StSh] pp. 70–81.$\big)$

`Tuesday 11.2` Application of Fourier series to PDEs: solving the 1-dimensional heat equation by the Fourier series. Setting of the problem with zero boundary values, and with convergence to the initial value in L^2, and enough smoothness in x or t as $t > 0$. Uniqueness of the solution. Writing the solution by Fourier series (in the x-variable), determining the coefficients as functions of t. Checking that the solution formula yields a solution (with smoothness for $t > 0$.) with the right initial and boundary values. ([StSh] pp. 18–20, 118–120.)

`Thursday 13.2` Some other examples of PDE:s (in the 2π-periodic case). Isoperimetric inequality (Queen Didi's problem) via Fourier series. ([StSh] pp. 101–105.)

`Monday 17.2` Finishing the isoperimetric inequality. Formal solution of the wave-equation by Fourier series. ([StSh] pp. 118–120, 6–18.)

`Tuesday 18.2` Discrete Fourier series (transform) in $\mathbb{Z}/N\mathbb{Z}$. Representation of given function by its finite Fourier series in this case. Parseval formula. Fast Fourier transform (FFT) and its computational cost. ([StSh] pp. 218-226.)

`Thursday 20.2` Last lecture: overview of Sobolev spaces via Fourier series. Coffee!

Content

Fourier analysis is a central tool in many areas of mathematics, including PDE:s, harmonic analysis, analytic number theory, mathematical physics, probability, etc...
The aim of the course 'Fourier analysis I' is to give basic and workable knowledge on Fourier series.
Lectures will follow somewhat freely the Finnish language notes[1] included on the webpage section 'Materials' .The actual content of the lectures can be seen by following the BLOG of the course below, where also corresponding pages of the following English text book will be referenced:

E.M. Stein & R. Shakarchi: *Fourier Analysis, An Introduction*, Princeton University Press, 2003

This book covers with enough precision the material covered during the lectures! Some more material for the interested students:

- L. Grafakos: *Classical Fourier Analysis*, Springer, 2008.
- Rudin, W.: *Real and Complex Analysis*, McGraw-Hill, Third ed. 1987;
  Rudin, W.: *Functional analysis*, McGraw-Hill, Second ed. 1990.

[1] The Finnish language lecture notes (covering both Fourier analysis I and II) are based on those produced by Kari Astala on fall 2012, that have been somewhat modified by later lecturers including the present one.