FOURIER ANALYSIS I. (Spring 2020)

6. EXERCISES (Thursday 27.2 14-16 in room C123)

1. Let a be a real constant. Show that the equation

$$u''(x) + au'(x) - u(x) = 0 \quad \text{for all} \quad x \in \mathbf{R}.$$

has no other 2π-periodic twice differentiable solutions than the trivial solution $u \equiv 0$.

[Hint: check what the equation tells about the Fourier series of u.]

2 Recall that in the lectures we described the movement of a (violin) string by the solution of the wave-equation

$$u_{tt}(x,t) = c^2 u_{xx}(x,t), \qquad u_t(x,0) = 0 \quad \text{and} \quad u(x,0) = f(x)$$

on the interval $(0,\pi)$ (with zero boundary values, initial velocity 0 and initial suspension $f(x)$) was given by the formula

$$u(x,t) = \sum_{n=1}^{\infty} c_n \cos(nct)\sin(nx) \qquad \text{with} \quad c_k = \frac{2}{\pi}\int_0^{\pi} \sin(ny)f(y)dy.$$

Try to argue that the equation is not smoothing the initial value at all as $t \to \infty$.

[Hint: what about periodicity in time...]

3. Let $f : [0, 2\pi) \to \mathbf{C}$ be a function. Given integer $N \geq 2$ define the function $F_N : Z(N) \to \mathbf{C}$ by letting

$$(*) \qquad F_N(k) := f(k2\pi/N) \qquad \text{for} \quad k = 0, 1, \ldots, N-1.$$

Compute the discrete Fourier coefficients of F_N (formula (8.2) of the Finnish lecture notes) in the case

$$f(x) = e^{ax}, \qquad x \in [0, 2\pi),$$

where $a > 0$ is a constant.

4. Let $f \in C_\sharp[0, 2\pi)$ and for given integer $N \geq 2$ define the function F_N as in formula $(*)$ of the previous exercise. Prove that for any fixed $n \geq 1$ we obtain the (standard) Fourier coefficients of f from the discrete ones of F_N in the following way:

$$\widehat{f}(n) = \lim_{N\to\infty} \widehat{F}_N(n).$$

Check this in an example by computing the standard Fourier transform of the function $f(x) = e^{ax}$ (with constant $a > 0$) and comparing with the result in exercise 3.

[Hint: Riemann sums for an integral are lurking behind...]

5. Suppose the Fourier series of a function $g \in C_{\#}(-\pi, \pi)$ is a lacunary series of the form

$$\sum_{k=-\infty}^{\infty} a_k e^{i2^{|k|}x}.$$

Show that then the partial Fourier sums are uniformly bounded, i.e. $|S_n g(x)| \leq C$ for some constant $C < \infty$ and for all $n \in \mathbf{N}$.

[Hint: Look at how the Fourier coefficients of the kernel $F_{2N} - F_N$ (difference of two different Fejer-kernels) look like. Make use of this and the basic properties (goodness) of Fejer-kernels.]

6**[1] Prove that the sequence $\left((2+\sqrt{3})^n\right)_{n=1}^{\infty}$ is *not* equidistributed (mod 1).

[1] These $*$-exercises are extras for afficinadoes, not required to get full points from exercises