Problem set 6, Advanced risk theory, 28.2.2020

Exercise 6.1. Let $\xi, \xi_1, \xi_2, \dots$ and $\eta, \eta_1, \eta_2, \dots$ be sequences of i.i.d. random variables such that

$$\mathbb{P}(\xi > x) = e^{-(x+2)}$$

for $x \geq 0$ and 1 otherwise and

$$\mathbb{P}(\eta > x) = x^{-2}$$

for large enough x. Suppose expectation $\mathbb{E}(\eta)$ exists. Assume further that $\mu = \mathbb{E}(\xi) = \mathbb{E}(\eta)$ and let $a > \mu$ be fixed.

Show that there exists a number $n_0 \in \mathbb{N}$ such that

$$\mathbb{P}(\xi_1 + \cdots + \xi_n > na) < \mathbb{P}(\eta_1 + \cdots + \eta_n > na)$$

for all $n \geq n_0$.

Exercise 6.2. Let Y'_n be as in Equation (4.4) of Section 4. Let $(D, \xi), (D_1, \xi_1), (D_2, \xi_2), \dots$ be i.i.d. random vectors. In particular, the variables concerning different years are independent, but the variables of the same year are not necessarily independent. We denote

$$Y'_\infty = \sum_{i=1}^{\infty} D_1 \dots D_{i-1} \xi_i.$$

Suppose Y'_∞ is well defined in the sense that the series converges almost surely. Let $U_0 > 0$ be the initial capital and $T(U_0) = \inf\{n \,|\, Y'_n > U_0\}$ the ruin time.

Show that

$$\mathbb{P}(T(U_0) < \infty) \geq \mathbb{P}(Y'_\infty > U_0).$$

Exercise 6.3 (Exercise 6.2 continues)**.** Show that

$$\mathbb{P}(Y'_\infty > U_0) \geq \mathbb{P}(T(U_0) < \infty)\mathbb{P}(Y'_\infty > 0).$$

Hint: You can start from equality $\mathbb{P}(Y'_\infty > U_0) = \sum_{n=1}^{\infty} \mathbb{P}(Y'_\infty > U_0, T(U_0) = n)$.

Exercise 6.4 (Exercise 6.2 continues)**.** Suppose $\xi_n = 1 - D_n$ for all n and $\mathbb{P}(D \in (0,1)) = 1$.

Calculate Y'_n for all n. Furthermore, determine Y'_∞ and

$$\lim_{U_0 \to \infty} \frac{\log \mathbb{P}(T(U_0) < \infty)}{\log U_0}.$$

Exercise 6.5 (Exercise 6.2 continues)**.** Assume, in addition, that the components of (D, ξ)-vectors are independent and suppose $\mathbb{E}(\xi^2) < \infty$ and $\mathbb{E}(D^2) < \infty$.

Explain why

$$Y'_\infty \overset{d}{=} \xi + DY'_\infty,$$

holds. On the right hand side, Y'_∞ is independent of the vector (ξ, D). Calculate $\mathbb{E}(Y'_\infty)$ and $\mathrm{Var}(Y'_\infty)$ using the moments of ξ and D.