Fourier Analysis I

Spring 2020

Homework 5

Exercise session: Thu 20 February, 14:15 - 16:00, Exactum C123; Stefanos Lappas, stefanos.lappas@helsinki.fi.

1. Denote $f(x) = ax + 1$ and $g(x) = x^2$. Determine coefficient a so that the L^2-distance $\|f - g\|_{L^2(-\pi,\pi)}$ is minimized. Try to make a geometric interpretation (in $L^2(-\pi, \pi)$) for the solution.

Proof. We have

$$\begin{aligned}
\|f - g\|^2_{L^2(-\pi,\pi)} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - g(x)|^2 dx \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} (x^2 - ax + 1)^2 dx \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} (x^4 + a^2x^2 + 1 - 2ax^3 - 2ax + 2x^2) dx \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} (x^4 - 2ax^3 + (a^2 + 2)x^2 - 2ax + 1) dx \\
&= \frac{1}{2\pi} \Big(\frac{2\pi^5}{5} + \frac{2(a^2 + 2)\pi^3}{3} + 2\pi \Big) \\
&= \frac{\pi^4}{5} + \frac{(a^2 + 2)\pi^2}{3} + 1.
\end{aligned}$$

Now, using standard calculus we see that the L^2-distance $\|f - g\|_{L^2(-\pi,\pi)}$ is minimized for $a = 0$.

Regarding the geometric interpretation we know that f is a line in the Hilbert space $L^2(-\pi, \pi)$. So in order to minimize the L^2-distance $\|f - g\|_{L^2(-\pi,\pi)}$ it is enough to form the orthogonal triangle ABC with vertices $A = x^2$, $B = ax + 1$ and $C = 1$. Here the triangle is orthogonal in B. Hence, by the Pythagoras' theorem we have:

$$\|x^2 - (ax + 1)\|^2_{L^2(-\pi,\pi)} + \|ax\|^2_{L^2(-\pi,\pi)} = \|x^2 - 1\|^2_{L^2(-\pi,\pi)}. \tag{1}$$

From (1) it follows that $a = 0$. □

2. Let $f \in C^1_\#$ and $\int_{-\pi}^{\pi} f(x)dx = 0$. Prove Poincare type inequality.

$$\int_{-\pi}^{\pi} |f(x)|^2 dx \;\leq\; \int_{-\pi}^{\pi} |f'(x)|^2 dx.$$

For which functions do you have equality here?

Proof. We apply Plancherel's formula to see that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2.$$

As $\widehat{f'}(n) = in\widehat{f}(n)$, we also have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f'(x)|^2 dx = \sum_{n=-\infty}^{\infty} |in\widehat{f}(n)|^2 = \sum_{n=-\infty}^{\infty} n^2|\widehat{f}(n)|^2.$$

We can then use the estimate $n^2 \geq 1$ for any $n \neq 0$ to get

$$\int_{-\pi}^{\pi} |f'(x)|^2 dx = 2\pi \sum_{n=-\infty}^{\infty} n^2|\widehat{f}(n)|^2 \geq 2\pi \sum_{n\neq 0} |\widehat{f}(n)|^2.$$

Because $\int_{-\pi}^{\pi} f(x)dx = 0$, we know that $\widehat{f}(0) = 0$. This means that

$$\int_{-\pi}^{\pi} |f'(x)|^2 dx \geq 2\pi \sum_{n\neq 0} |\widehat{f}(n)|^2 = 2\pi \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2 = \int_{-\pi}^{\pi} |f(x)|^2 dx.$$

For the equality to hold, we must have $n^2|\widehat{f}(n)|^2 = |\widehat{f}(n)|^2$ for all n. This in particular means that $\widehat{f}(n) = 0$ whenever $|n| \neq 1$. So the equality can only hold when $f(x) = \widehat{f}(1)e^{ix} + \widehat{f}(-1)e^{-ix} \iff f(x) = a\sin(x) + b\cos(x)$ for $a, b \in \mathbf{C}$. We see that for any such function f the equality does indeed hold. □

3. Let $f \in L^2(-\pi, \pi)$. Find the trigonometric polynomial $p(x) := \sum_{n=-N}^{N} c_n e^{inx}$ which is closest to f in L^2-norm, i.e. find the coefficients c_n that minimise the quantity

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left| f(x) - \sum_{n=-N}^{N} c_n e^{inx} \right|^2 dx$$

Proof. Using Plancherel's formula we see that if we denote $c_n = 0$ for any $|n| > N$, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Big|f(x) - \sum_{n=-N}^{N} c_n e^{inx}\Big|^2 dx = \sum_{|n|\le N} |\widehat{f}(n) - c_n|^2 + \sum_{|n|>N} |\widehat{f}(n)|^2 \ge \sum_{|n|>N} |\widehat{f}(n)|^2.$$

The equality holds if $|\widehat{f}(n) - c_n| = 0$ for any $|n| \le N$, in other words when $c_n = \widehat{f}(n)$. So the closest trigonometric polynomial in L^2-norm is the partial sum of the Fourier series. □

4. Suppose $f \in C^1_{\#}(-\pi, \pi)$. Show that the Fourier series of f converges absolutely, i.e. we have $\sum |\widehat{f}(n)| < \infty$.

Proof. Suppose $f \in C^1_{\#}(-\pi, \pi)$. Then the Fourier coefficients of f' are well-defined and for all $n \in \mathbf{Z}$ we have the formula

$$\widehat{f'}(n) = in\widehat{f}(n).$$

We now apply the Cauchy-Schwartz inequality to the two sequences

$$(1/n)_{n=1}^{\infty} \quad \text{and} \quad (|\widehat{f'}(n)|)_{n=1}^{\infty}.$$

The first one is obviously in l^2, and the second one is too since by Plancherel's formula

$$\sum_{n=-\infty}^{\infty} |\widehat{f'}(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f'(x)|^2 dx < \infty.$$

Thus we have that

$$\begin{aligned}\sum_{n=1}^{\infty} |\widehat{f}(n)| &= \sum_{n=1}^{\infty} \frac{1}{n} |\widehat{f'}(n)| \\ &\le \Big(\sum_{n=1}^{\infty} \frac{1}{n^2}\Big)^{1/2} \Big(\sum_{n=1}^{\infty} |\widehat{f'}(n)|^2\Big)^{1/2} \\ &< \infty.\end{aligned}$$

Similarly we see that

$$\sum_{n=-\infty}^{-1} |\widehat{f}(n)| < \infty.$$

Hence

$$\sum_{n=-\infty}^{\infty} |\widehat{f}(n)| = \sum_{n=-\infty}^{-1} |\widehat{f}(n)| + |\widehat{f}(0)| + \sum_{n=1}^{\infty} |\widehat{f}(n)| < \infty.$$

Thus the Fourier series of f converges absolutely. □

5. Solve – to find a formal solution formula is enough – by using Fourier series the following PDEs. Here $x \in [0, 2\pi)$, $t \geq 0$ and $x \to u(x,t)$ is assumed to be 2π-periodic, and the solution satisfies the initial value $u(x,0) = f(x)$, where $f \in L^2(-\pi,\pi)$ (of course you may assume that f is 2π-periodic). Instead of sine series, since we are in the 2π-periodic case, use just standard Fourier series to make the 'Ansatz' $u(x,t) = \sum_{n\in\mathbf{Z}} A_n(t)e^{inx}$.

(i) $\frac{d}{dt}u(x,t) = -\left(\frac{d}{dx}\right)^4 u(x,t)$.

(ii) $\frac{d}{dt}u(x,t) = \frac{d}{dx}u(x,t)$.

Try to guess (no rigorous reasoning needed here) in which one of the previous equations the solution is always smooth in the variable x for $t > 0$.

Proof. (i) We assume that our solution can be written as $u(x,t) = \sum_{n\in\mathbf{Z}} A_n(t)e^{inx}$. Then for the first equation we derive

$$A_n'(t) = -n^4 A_n(t). \tag{2}$$

The solution of (2) is given by

$$A_n(t) = c_n e^{-n^4 t}.$$

Using the initial condition $u(x,0) = f(x)$ we find that $c_n = \widehat{f}(n)$. Hence,

$$u(x,t) = \sum_{n\in\mathbf{Z}} \widehat{f}(n)e^{-n^4 t}e^{inx}.$$

(ii) We assume again that our solution is of the form $u(x,t) = \sum_{n\in\mathbf{Z}} A_n(t)e^{inx}$. Then from the second equation we get

$$A_n'(t) = inA_n(t). \tag{3}$$

The solution of (3) is given by

$$A_n(t) = c_n e^{int}.$$

Using the initial condition $u(x,0) = f(x)$ we find that $c_n = \widehat{f}(n)$. Thus,

$$u(x,t) = \sum_{n\in\mathbf{Z}} \widehat{f}(n)e^{int}e^{inx}.$$

□

6. Compute the Fourier series of $f(x) = x^2$, $x \in (-\pi, \pi)$ and compute the L^2-norm of f in two ways: first by direct computation and then using the Fourier-coefficients. Use this to compute the $\sum_{n=1}^{\infty} n^{-4}$.

Proof. A direct computation shows that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^4 dx = \frac{\pi^4}{5}.$$

Next, we recall from exercise sheet 4 the Fourier coefficients $\widehat{f}(n)$. We have

$$\widehat{f}(n) = \begin{cases} \frac{\pi^2}{3}, & \text{if } n = 0 \\ \frac{2}{n^2}(-1)^n, & \text{if } n \neq 0. \end{cases}$$

Using Plancherel's formula, we know that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2.$$

This means that we have

$$\frac{\pi^4}{5} = \left(\frac{\pi^2}{3}\right)^2 + \sum_{n\neq 0} \left|\frac{2(-1)^n}{n^2}\right|^2 = \frac{\pi^4}{9} + 2\sum_{n=1}^{\infty} \frac{4}{n^4}.$$

We can now solve that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\frac{\pi^4}{5} - \frac{\pi^4}{9}}{8} = \frac{\pi^4}{90}.$$

□

7. Try to analyse rigorously the behaviour of the solutions u in problem 5. Especially, could you write a simpler solution formula (than using Fourier series) for the solution in part (ii) ?

Proof. Recall that a sequence of complex numbers $\{x_n\}_{n=-\infty}^{\infty}$ is a sequence of rapid decay if:

$$\sum_{n=-\infty}^{\infty} |n|^k |x_n| < \infty \qquad \forall k \in \mathbb{N} \cup \{0\}.$$

Now, we know that the function $f \in L^1(-\pi, \pi)$ is smooth if and only if the sequence of its Fourier coefficients $\{\widehat{f}(n)\}_{n=-\infty}^{\infty}$ is a sequence of rapid decay. In part (i) of problem 5

we see that our solution is smooth in the variable x for $t > 0$ since $e^{-n^4 t} \to 0$ extremely quickly. In part (ii) the regularity of our solution remains the same, i.e. our solution is not smooth in the variable x for $t > 0$.
A simpler solution formula for the solution in part (ii) without using Fourier series is given by

$$u(x,t) = f(x+t),$$

since $\widehat{f}(x+t)(n) = \widehat{f}(n)e^{int}$. $\square$