Algebra II

Exercise 6 (27.2.2020)

1. In the lectures it was proved, using the Sylow theorems, that any group of order 30 has a non-trivial normal subgroup (that is, a normal subgroup which is not the trivial group or the whole group). It can be proved that any group of order 30 is isomorphic to one of the following: $\mathbb{Z}_{30},\ \mathbb{Z}_5 \times S_3,\ \mathbb{Z}_3 \times D_{10},\ D_{30}$. Give at least one example of a non-trivial normal subgroup for each of these groups.

2. Suppose that G, G' are groups and $f\colon G \to G'$ a homomorphism. Suppose that $H \trianglelefteq G$ and $H' \trianglelefteq G'$. Prove the following:

a) $f^{-1}(H') \trianglelefteq G$.

b) If f is surjective, then $f(H) \trianglelefteq G'$. Give an example which shows that the surjectivity assumption is necessary here.

(You may assume known that $f^{-1}(H') \leq G$ and $f(H) \leq G'$.)

3. Prove that no group of order 80 is simple.

4. Prove that every group of order 33 is isomorphic to $\mathbb{Z}_{33}$.

5. Consider the group $Gl(2, \mathbb{C})$ of all invertible (2×2)-matrices with complex coefficients, with matrix multiplication. Denote the unit matrix by I. Let G be the subgroup of this group, generated by the matrices

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

a) Prove by computing that $A^4 = B^4 = I$, $A^2 = B^2$ and $BA = A^3B$.

b) Using the equations of item a), deduce that every element in the group G can be presented as a product $A^{k_1}B^{l_1} \cdots A^{k_n}B^{l_n}$, where the exponents are all non-negative. Prove then that every element of G can be written in the form A^iB^j, where $i, j \in \mathbb{N}$. Finally, prove that with different exponents i, j we only obtain finitely many different elements (that is, the group G is finite).

c) Give a list of the elements of G.

(Additional information: The group G is called the *quaternion group*. It can be proved that every group of order 8 is isomorphic to one of the following: $\mathbb{Z}_8,\ \mathbb{Z}_4 \times \mathbb{Z}_2,\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2,\ D_8$ or the quaternion group.)

6. We prove that (up to isomorphism) there exist exactly two groups which have order 6.
Suppose $G = \{e, a, b, c, d, f\}$ is a group. By the Cauchy Theorem G has subgroups of order 3 and 2. The subgroup of order 3 is normal, since it has index 2. Suppose that $\{e, a, b\} \trianglelefteq G$ and $\{e, c\} \leq G$. Prove (for example by constructing the multiplication tables) that

a) if $cac^{-1} = a$, then $G \cong \mathbb{Z}_6$.

b) if $cac^{-1} \neq a$, then $G \cong S_3$. (Remark. $S_3 \cong D_6$)

(Additional fact: It can be proved that if p is a prime, $p \geq 3$, then there exist (up to isomorphism) exactly two groups of order $2p$, namely $\mathbb{Z}_{2p}$ and D_{2p}.)