Problem I. Calculations[32pt]

1[8pt]. Calculate the following limit:

$$\lim_{x\to\infty} \frac{e^x+1}{e^x-1}.$$

Solution. The limit is

$$\lim_{x\to\infty} \frac{e^x+1}{e^x-1} = 1.$$

□

2[8pt]. Calculate the following limit:

$$\lim_{x\to 0} \frac{\ln(1+x)}{\sin(x)}.$$

Solution. The limit is

$$\lim_{x\to 0} \frac{\ln(1+x)}{\sin(x)} = 1.$$

□

3[8pt]. Calculate the derivative of the following function:

$$x \in (-\frac{\pi}{2}, \frac{\pi}{2}), \quad x \mapsto \ln(\cos(x)).$$

Solution. The derivative is

$$x \in (-\frac{\pi}{2}, \frac{\pi}{2}), \quad x \mapsto -\tan(x).$$

□

4[8pt]. Calculate the derivative of the following function:

$$x \in (-1,1), \quad x \mapsto \arccos(x).$$

Solution. The derivative is

$$x \in (-1,1), \quad x \mapsto -\frac{1}{\sqrt{1-x^2}}.$$

□

Problem II. Study of a function[68pt]

In this exercise we study the function $F : \mathbb{R}_{>0} \to \mathbb{R}$ defined as

$$F(x) = \frac{\sin(x)}{x}.$$

Before you start[5pt]. Try to draw a rough graph of the function F.

Solution. It is recommended to mark some special values, limit at infinity etc.; positions of zeros and important tangents will also be appreciated. □

Part I[8pt]. In this part we study some basic properties of F.

1[2pt]. Recall that sin is derivable on $\mathbb{R}_{>0}$. Calculate the first and second derivatives of sin.

Solution. We have $(\sin)'(x) = \cos(x)$ and $(\cos)'(x) = -\sin(x)$ for all $x \in \mathbb{R}_{>0}$. □

2[2pt]. Use $(\sin)'(0)$ to determine the following limit (alternatively l'Hôpital's rule is allowed):

$$\lim_{x\to 0} F(x).$$

Solution. By definition,

$$(\sin)'(0) = \lim_{x\to 0} \frac{\sin(x) - \sin(0)}{x - 0} = \lim_{x\to 0} F(x).$$

Since $(\sin)'(0) = \cos(0) = 1$, $\lim\limits_{x\to 0} F(x)$ exists and $\lim\limits_{x\to 0} F(x) = 1$. □

3[2pt]. Calculate the derivative of F on $\mathbb{R}_{>0}$.

Solution. We apply the quotient rule: for all $x \in \mathbb{R}_{>0}$, $x \neq 0$ and

$$F'(x) = \frac{x \cdot (\sin)'(x) - \sin(x) \cdot 1}{x^2} = \frac{x\cos(x) - \sin(x)}{x^2}.$$

□

4[2pt]. Show that F' on $\mathbb{R}_{>0}$ has the same sign as $h(x) = x\cos(x) - \sin(x)$.

Solution. By the question above we have that, on $\mathbb{R}_{>0}$,

$$x^2 \cdot F'(x) = h(x).$$

Since $x^2 \geq 0$, we conclude that F' and h have the same sign on $\mathbb{R}_{>0}$. □

Part II[9pt]**.** In this part we study some properties of F on the interval $[\pi, 2\pi]$.

$1^{[1pt]}$. Show that $F(\pi) = F(2\pi) = 0$.

Solution. Since $\sin(\pi) = \sin(2\pi) = 0$, dividing by non-zero quantities yields $F(\pi) = F(2\pi) = 0$. □

$2^{[2pt]}$. Show that $F'(c) = 0$ for some $c \in (\pi, 2\pi)$.

Solution. The function F is continuous on $[\pi, 2\pi]$ and derivable on $(\pi, 2\pi)$, and furthermore the question above shows that $F(\pi) = F(2\pi)$. Applying Rolle's theorem yields existence of some $c \in (\pi, 2\pi)$ such that $F'(c) = 0$. □

$3^{[2pt]}$. Prove that F is bounded on $[\pi, 2\pi]$ and that its minimum on $[\pi, 2\pi]$ is attained.

Solution. The function F is continuous on the closed bounded interval $[\pi, 2\pi]$. Applying Weierstrass' theorem, we know that F is bounded on $[\pi, 2\pi]$ and its minimum (and also its maximum) is attained on $[\pi, 2\pi]$. □

$4^{[2pt]}$. Show that F is negative on $[\pi, 2\pi]$. Find the maximum of F on $[\pi, 2\pi]$.

Solution. On $[\pi, 2\pi]$, F has the same sign as the sin function. We know that $\sin \leq 0$ on $[\pi, 2\pi]$ so that $F \leq 0$ on $[\pi, 2\pi]$. This shows that $\sup\limits_{x \in [\pi, 2\pi]} F(x) \leq 0$. Since we have shown that $F(\pi) = 0$, we conclude that $\sup\limits_{x \in [\pi, 2\pi]} F(x) = 0$ and this supremum is a maximum (it is attained at $\pi \in [\pi, 2\pi]$). □

$5^{[2pt]}$. [$\star$] Justify that the minimum of F on $[\pi, 2\pi]$ is attained only once.

Solution. Suppose that the minimum of F is attained two times at points $x, y \in [\pi, 2\pi]$. First x, y must be in $(\pi, 2\pi)$ since $F(\pi) = F(2\pi) = 0$ and F takes negative values between π and 2π. Now F is derivable on $[\pi, 2\pi]$ so that Rolle's theorem implies that $F'(x) = F'(y) = 0$ in such a way that $h(x) = h(y) = 0$. Applying Rolle's theorem again for the smooth function h, we know that $h'(c) = 0$ for some c between x and y, in particular for some $c \in (\pi, 2\pi)$. A calculation shows that $h'(x) = -x \cdot \sin(x)$ so that h' is never 0 on the interval $(\pi, 2\pi)$: this contradiction shows that the minimum of F on $[\pi, 2\pi]$ is attained only once. □

Part III[5pt]**.** In this part we study the solutions to $F(x) = 0$ on $\mathbb{R}_{>0}$.

$1^{[1pt]}$. Find the smallest solution a_1 to $F(x) = 0$ on $\mathbb{R}_{>0}$.

Solution. On $\mathbb{R}_{>0}$, $F(x) = 0$ if and only if $\sin(x) = 0$. The smallest solution to $\sin(x) = 0$ on $\mathbb{R}_{>0}$ is $x = \pi$, so $a_1 = \pi$ is the smallest solution to $F(x) = 0$ on $\mathbb{R}_{>0}$. □

$2^{[1pt]}$. Show that the second smallest solution a_2 to $F(x) = 0$ on $\mathbb{R}_{>0}$ is $a_2 = 2\pi$.

Solution. By the same argument as before, the second smallest solution $a_2 = 2\pi$. □

$3^{[1pt]}$. All solutions to $F(x) = 0$ on $\mathbb{R}_{>0}$ form an increasing sequence $\{a_k\}_{k\geq 1}$. What is a_k?

Solution. It is easy to iterate the argument before to see that $a_k = k\pi$ for all $k \geq 1$. □

$4^{[2pt]}$. Is the sequence $\{a_k\}_{k\geq 1}$ bounded? Is the sequence $\{a_k\}_{k\geq 1}$ convergent?

Solution. The sequence $\{a_k\}$ is not bounded (Archimedean property). Since any convergent sequence is bounded, by contraposition this shows that the sequence $\{a_k\}$ does not converge. □

Part IV$^{[11pt]}$. In this part we study the solutions to $F'(x) = 0$ on $\mathbb{R}_{>0}$. Let $h(x) = x\cos(x) - \sin(x)$.

$1^{[1pt]}$. Determine the sign of $h(a_k)$ at the points a_k in terms of $k \geq 1$.

Solution. Recall that for all $k \in \mathbb{Z}_{>0}$, $a_k = k\pi$ and $\cos(k\pi) = (-1)^k$, $\sin(k\pi) = 0$. We deduce that for all $k \in \mathbb{Z}_{>0}$, $h(a_k) = (-1)^k \cdot k\pi$. The quantity $h(a_k)$ is strictly positive if k is even and strictly negative if k is odd. □

$2^{[2pt]}$. Prove that for all $k \geq 1$, the equation $h(x) = 0$ has a solution on (a_k, a_{k+1}).

Solution. The function h is continuous on the interval $[a_k, a_{k+1}]$ and $h(a_k) \cdot h(a_{k+1}) < 0$ (since they have different signs and are not zero). By Bolzano's theorem, the equation $h(x) = 0$ has at least one solution $c \in (a_k, a_{k+1})$. □

$3^{[2pt]}$. Calculate the derivative of h on $\mathbb{R}_{>0}$.

Solution. Applying the product rule, $h'(x) = \cos(x) - x \cdot \sin(x) - \cos(x) = -x \cdot \sin(x)$ on $\mathbb{R}_{>0}$. □

$4^{[1pt]}$. For all $k \geq 1$, study the sign of h' on (a_k, a_{k+1}).

Solution. On the interval (a_k, a_{k+1}), from the previous question we know that h' has the same sign as the function $-\sin$. From this we deduce that on (a_k, a_{k+1}) with $k \in \mathbb{Z}_{>0}$, h' is strictly positive if k is odd and h' is strictly negative if k is even. □

$5^{[2pt]}$. Conclude that, for all $k \geq 1$, the solution to the equation $h(x) = 0$ on (a_k, a_{k+1}) is unique.

Solution. The uniqueness of the solution follows from the strict monotonicity of h on the interval (a_k, a_{k+1}) proven in the previous question. More precisely, suppose that $h(x) = h(y) = 0$ with $x, y \in (a_k, a_{k+1})$ and $x < y$. Since the function h is smooth on $[a_k, a_{k+1}]$, Rolle's theorem shows the existence of $c \in (x, y) \subset (a_k, a_{k+1})$ such that $h'(c) = 0$. This contradicts the previous question since h' is non-zero on (a_k, a_{k+1}) for any $k \in \mathbb{Z}_{>0}$. □

$6^{[1pt]}$. Show that the equation $F'(x) = 0$ has a unique solution b_k on (a_k, a_{k+1}).

Solution. Recall that for all $k \in \mathbb{Z}_{>0}$, we have $x^2 \cdot F'(x) = h(x)$ on (a_k, a_{k+1}). It follows from the previous question that the solution to the equation $F'(x) = 0$ is the same as the one to the equation $h(x) = 0$ on (a_k, a_{k+1}): in particular it is unique and we denote it by b_k. □

$7^{[2pt]}$. Is the sequence $\{b_k\}_{k\geq 1}$ increasing? Is the sequence $\{b_k\}_{k\geq 1}$ bounded?

Solution. For all $k \in \mathbb{Z}_{>0}$, we have $b_k \in (a_k, a_{k+1})$. We deduce that $b_k < a_{k+1} < b_{k+1}$ for all $k \in \mathbb{Z}_{>0}$ so that the sequence $\{b_k\}$ is strictly increasing. Also, for all $k \in \mathbb{Z}_{>0}$ we have that $a_k < b_k$, and since the sequence $\{a_k\}$ is not bounded from above, by the comparaison theorem, $\{b_k\}$ is not bounded from above neither: in particular this shows that $\{b_k\}$ is not a bounded sequence. □

Part V$^{[4pt]}$. We study some finer properties of the sequences $\{a_k\}$ and $\{b_k\}$.

$1^{[2pt]}$. Prove that for all $k \geq 1$, $b_k \in (a_k, a_k + \frac{\pi}{2})$.

Solution. We need to improve one question before: we study the sign of $h(a_k) \cdot h(a_k + \frac{\pi}{2})$. For all $k \in \mathbb{Z}_{>0}$, we have $h(a_k + \frac{\pi}{2}) = 0 - \sin(k\pi + \frac{\pi}{2})$, so that $h(a_k + \frac{\pi}{2})$ is strictly negative if k is even and strictly positive if k is odd. By a question before, we conclude that for all $k \in \mathbb{Z}_{>0}$, $h(a_k) \cdot h(a_k + \frac{\pi}{2}) < 0$. The function h is continuous on the interval $[a_k, a_k + \frac{\pi}{2}]$, it follows from Bolzano's theorem that the equation $h(x) = 0$ has a solution $b'_k \in (a_k, a_k + \frac{\pi}{2})$. Since the solution to the equation $h(x) = 0$ is unique on the interval (a_k, a_{k+1}), we know that $b'_k = b_k$ and consequently, $b_k \in (a_k, a_k + \frac{\pi}{2})$. □

$2^{[2pt]}$. Determine and justify rigorously the limit

$$\lim_{k\to\infty} \frac{b_k}{a_k}.$$

Proof. For all $k \in \mathbb{Z}_{>0}$, we have that $a_k < b_k < a_{k+1}$ and since $a_k = k\pi > 0$, we have the inequalities $1 = \frac{a_k}{a_k} < \frac{b_k}{a_k} < \frac{a_{k+1}}{a_k} = \frac{k+1}{k}$. Now the sequences $\{1\}$ and $\{\frac{k+1}{k}\}$ both converges and have the same limit (namely 1), by the squeeze theorem, $\lim\limits_{k\to\infty} \frac{b_k}{a_k}$ exists and $\lim\limits_{k\to\infty} \frac{b_k}{a_k} = 1$. □

Part VI$^{[4pt]}$. In this part we study some properties of F on the interval $(0, \frac{\pi}{2}]$.

$1^{[2pt]}$. Prove that F is strictly decreasing on $(0, \frac{\pi}{2}]$.

Solution. We study the sign of F' on $(0, \frac{\pi}{2})$. By a question before, this is the same as studying the sign of the function h on $(0, \frac{\pi}{2})$. Recall that $h'(x) = -x \cdot \sin(x)$ so that on $(0, \frac{\pi}{2})$, h is strictly decreasing. Furthermore, $h(0) = 0$ so that on $(0, \frac{\pi}{2})$, the function h takes strictly negative values. It follows that F' is strictly decreasing on the interval $(0, \frac{\pi}{2})$. □

2[2pt]. Justify that

$$\forall x \in (0, \frac{\pi}{2}], \quad \frac{2}{\pi} \leq F(x) < 1.$$

Solution. By the question before, F is strictly decreasing on the interval $(0, \frac{\pi}{2})$. Furthermore, we have shown that $\lim_{x\to 0} F(x) = 1$. It follows that the range of F on the interval $(0, \frac{\pi}{2}]$ is bounded by $\lim_{x\to\frac{\pi}{2}} F(x)$ and $\lim_{x\to 1} F(x)$, i.e. respectively $\frac{2}{\pi}$ and 1.

To show the strictly inequality for the right hand side, we need to show that the supremum 1 is not attained on $(0, \frac{\pi}{2}]$. By contradiction, it $F(x_0) \geq 1$ for some $x_0 > 0$, then the strict monotonicity of F shows that for all $x \in (0, x_0)$, $F(x) > 1$. This is a contradiction since the supremum of F on $(0, \frac{\pi}{2})$ is 1 by the above discussion. □

Part VII[9pt]. In this part we study the extrema of F on $\mathbb{R}_{>0}$.

1[2pt]. Determine the limit

$$\lim_{x\to\infty} F(x).$$

Solution. Since $-1 \leq \sin(x) \leq 1$, we have that $-\frac{1}{x} \leq F(x) \leq \frac{1}{x}$ on $\mathbb{R}_{>0}$. Since $\lim_{x\to\infty} -\frac{1}{x} = \lim_{x\to\infty} \frac{1}{x} = 0$, it follows from the squeeze theorem that $\lim_{x\to\infty} F(x) = 0$. □

2[3pt]. Prove that $\sin(x) < x$ on $\mathbb{R}_{>0}$.

Solution. Consider the function $l(x) = x - \sin(x)$. This function is smooth on $\mathbb{R}$, $l(0) = 0$ and $l'(x) = 1 - \cos(x) \geq 0$ so that l is increasing on $\mathbb{R}$ (and in particular $\mathbb{R}_{>0}$). It follows (by the mean value theorem) that $l \geq 0$ on $\mathbb{R}_{>0}$.

To prove the strictly inequality one needs an extra argument. Indeed, if for some $x_0 > 0$, $\sin(x_0) = x_0$ then necessarily $\sin(y) = y$ for all $y \in (0, x_0)$. This implies that the function l has derivative constant equal to 0 on some interval $(0, x_0)$, and this is impossible for $y \in (0, x_0)$ small enough by the relation $l'(x) = 1 - \cos(x)$. □

3[2pt]. Determine the supremum of F on $\mathbb{R}_{>0}$.

Solution. Since $\lim_{x\to 0} F(x) = 1$, we know that $\sup_{\mathbb{R}_{>0}} F \geq 1$. Also, from the question above, we know that on $\mathbb{R}_{>0}$, we have $F < 1$ so that $\sup_{\mathbb{R}_{>0}} F \leq 1$. Together, the supremum of F on $\mathbb{R}_{>0}$ is 1. □

4[2pt]. Prove that the minimum of F on $\mathbb{R}_{>0}$ is attained on the interval (a_1, a_2).

Solution. Since F is everywhere continuous, the minimum of F on the interval $[a_1, a_2]$ is attained at some point $c \in [a_1, a_2]$ by Weierstrass' theorem. Furthermore, $c \neq a_1$ and $c \neq a_2$. Indeed, $F(a_1) = F(a_2) = 0$ by definition and F takes negative values on (a_1, a_2). In particular, $F(c) < 0$.

We now show that the infimum of F on $\mathbb{R}_{>0}$ is $F(c)$ by showing that for all $x \in \mathbb{R}_{>0}$, $F(x) \geq F(c)$. We distinguish two cases:

- If $x \in [a_k, a_{k+1}]$ with k an even integer, then $F(x) \geq 0$ so that $F(x) > F(c)$.
- If $x \in (a_k, a_{k+1})$ with k an odd integer, then writing $k = 2m+1$ we have $F(x) < 0$ and $F(x) = \frac{\sin(x)}{x} = \frac{\sin(x-2m\pi)}{x} \geq \frac{\sin(x-2m\pi)}{x-2m\pi} = F(x-2m\pi)$. Since $x - 2m\pi \in (a_1, a_2)$, we have that $F(x-2m\pi) \geq F(c)$ and consequently, $F(x) \geq F(c)$. □

Before you finish$^{[5pt]}$**.** Draw a better graph of the function F.

Solution. It is advisory to include/mark the information gathered in the previous questions. □

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Extra part$^{[8pt]}$**.** We study some finer asymptotics of the sequence $\{b_k\}$.

1$^{[2pt]}$. Find a relation between $\arctan(b_k)$ and b_k for all $k \in \mathbb{Z}_{>0}$.

Solution. Since b_k is a solution to the equation $h(x) = 0$, we have that $b_k \cdot \cos(b_k) = \sin(b_k)$, i.e. $\tan(b_k) = b_k$. To get a relation between $\arctan(b_k)$ and b_k, use the fact that $b_k \in (k\pi, (k+1)\pi)$ so that $\arctan(b_k) = b_k - k\pi$. □

2$^{[2pt]}$. Prove that

$$\lim_{x \to 0} \frac{\arctan(x) - x}{x^3} = -\frac{1}{3}.$$

Solution. One can use for instance l'Hôpital's rule:

$$\lim_{x \to 0} \frac{\arctan(x) - x}{x^3} = \lim_{x \to 0} \frac{\frac{1}{1+x^2} - 1}{3x^2} = \lim_{x \to 0} \frac{-\frac{1}{1+x^2}}{3} = -\frac{1}{3}.$$

where we have used the fact that $(\arctan)'(x) = \frac{1}{1+x^2}$. □

3$^{[2pt]}$. Prove that

$$\lim_{k \to \infty} k \cdot \left[\left(a_k + \frac{\pi}{2}\right) - b_k\right] = \frac{1}{\pi}.$$

Solution. A first observation is that $\lim\limits_{k \to \infty} [(a_k + \frac{\pi}{2}) - b_k] = 0$. Indeed, since b_k diverges to plus infinity, by the relation $\tan(b_k) = b_k$ we know that $\tan(b_k)$ also diverges to infinity so that the sequence $b_k - k\pi = \arctan(b_k)$ converges to $\frac{\pi}{2}$.

This is probably not the best argument but one possibility is to consider the quantity

$$\lim_{k \to \infty} \frac{(-1)^k}{k} \frac{h(a_k + \frac{\pi}{2}) - h(b_k)}{(a_k + \frac{\pi}{2}) - b_k}.$$

On the one hand, by considering $\frac{(-1)^k}{k} h'(a_k + \frac{\pi}{2})$, the limit is $-\pi$. On the other hand it can also be written as

$$\lim_{k\to\infty} -\frac{1}{k \cdot [(a_k + \frac{\pi}{2}) - b_k]}.$$

Identifying we get the desired identity. □

$4^{[2pt]}$. Determine the following limit

$$\lim_{k\to\infty} k^2 \cdot \left[\left(a_k + \frac{\pi}{2}\right) - b_k - \frac{1}{k\pi}\right].$$

Solution. We propose a more general approach. Notice first that since $b_k > 0$ for all $k \in \mathbb{Z}_{>0}$,

$$\arctan\left(\frac{1}{b_k}\right) = \frac{\pi}{2} - \arctan(b_k) = \frac{\pi}{2} - (b_k - a_k) = a_k + \frac{\pi}{2} - b_k.$$

Applying a question before to the sequence $\{\frac{1}{b_k}\}$ which converges to 0, we have

$$\lim_{k\to\infty} \frac{\arctan\left(\frac{1}{b_k}\right) - \frac{1}{b_k}}{(b_k)^{-3}} = -\frac{1}{3}.$$

This shows that

$$(b_k)^3 \cdot \left[\left(a_k + \frac{\pi}{2}\right) - b_k - \frac{1}{b_k}\right]$$

converges to $-\frac{1}{3}$. In particular we have

$$k^2 \cdot \left[\left(a_k + \frac{\pi}{2}\right) - b_k - \frac{1}{b_k}\right]$$

converges to 0. Thus

$$\lim_{k\to\infty} k^2 \cdot \left[\left(a_k + \frac{\pi}{2}\right) - b_k - \frac{1}{k\pi}\right] = \lim_{k\to\infty} k^2 \cdot \left[\frac{1}{k\pi} - \frac{1}{b_k}\right].$$

But

$$\lim_{k\to\infty} k^2 \cdot \left[\frac{1}{k\pi} - \frac{1}{b_k}\right] = \lim_{k\to\infty} k^2 \cdot \left[\frac{b_k - k\pi}{k\pi \cdot b_k}\right] = \frac{-\frac{\pi}{2}}{\pi \cdot \pi} = -\frac{1}{2\pi}.$$

Finally,

$$\lim_{k\to\infty} k^2 \cdot \left[\left(a_k + \frac{\pi}{2}\right) - b_k - \frac{1}{k\pi}\right] = -\frac{1}{2\pi}.$$

Arguments of this type can be made systematic using the theory of Taylor expansions. □