Algebra II. Exercise 6.
Solutions.

1. Some examples of non-trivial normal subgroups:

- $\mathbb{Z}_{30}$: every subgroup of the form $\langle [n]_{30} \rangle$, where $n|30$.
- $\mathbb{Z}_5 \times S_3$: for example $\mathbb{Z}_5 \times \{(1)\}$, $\{[0]_5\} \times S_3$, $\{[0]_5\} \times A_3$, $\mathbb{Z}_5 \times A_3$.
- $\mathbb{Z}_3 \times D_{10}$: for example $\mathbb{Z}_3 \times \{e_{D_{10}}\}$, $\{[0]_3\} \times D_{10}$, $\{[0]_3\} \times \langle \rho_5 \rangle$, $\mathbb{Z}_3 \times \langle \rho_5 \rangle$.
- D_{30}: for example $\langle \rho_{15} \rangle$.

Here ρ_n denotes the cycle $(1\ 2\ \ldots\ n)$.
We used the following fact: If G, H are groups and $K \trianglelefteq G$, $N \trianglelefteq H$, then $K \times N \trianglelefteq G \times H$. Proof: Let $(g, h) \in G \times H$ and $(k, n) \in K \times N$. Then $(g, h)(k, n)(g, h)^{-1} = (g, h)(k, n)(g^{-1}, h^{-1}) = (gkg^{-1}, hnh^{-1}) \in K \times N$. In the last step we used the assumptions $K \trianglelefteq G$, $N \trianglelefteq H$.

2. a) Let $g \in G$ and $h \in f^{-1}(H')$. We claim that $ghg^{-1} \in f^{-1}(H')$. Now

$$f(ghg^{-1}) = f(g)f(h)f(g^{-1}) = f(g)f(h)f(g)^{-1},$$

which is an element of H', since $f(h) \in H'$ and $H' \trianglelefteq G'$. This proves the claim.
b) Let $g' \in G'$ and $h' \in f(H)$. We claim that $g'h'g'^{-1} \in f(H)$. Choose $h \in H$ such that $f(h) = h'$. Since f is surjective, we can also choose $g \in G$, such that $f(g) = g'$. Since $H \trianglelefteq G$, we have that $ghg^{-1} \in H$. From this it follows that $g'h'g'^{-1} = f(g)f(h)f(g)^{-1} = f(ghg^{-1}) \in f(H)$. This proves the claim.
To show that the assumption of surjectivity is necessary, consider any group G having a subgroup H, which is not normal. Then the inclusion $\iota \colon H \to G$ is a homomorphism, and $H \trianglelefteq H$, but $\iota(H) = H$ is not normal in G.

3. Suppose that G is a group, $|G| = 80 = 5 \cdot 2^4$.
Consider first the Sylow 5-subgroups. The number of those, s_5, is a factor of 16 and $s_5 \equiv 1 \pmod 5$. The factors of 16 are $1, 2, 4, 8$ and 16, and of these

only 1 and 16 are congruent to 1 modulo 5. So we notice that the possible values for s_5 are 1 and 16.
(i) If $s_5 = 1$, this is the only Sylow subgroup, so it is normal, and we are done.
(ii) If $s_5 = 16$, we first notice that the intersection of two different 5-subgroups is necessarily trivial, so the 5-subgroups contain $16 \cdot 4 + 1 = 65$ different elements. Then consider the Sylow 2-subgroups (which are of order 16). Again we notice that the intersection of a 2-subgroup and a 5-subgroup is necessarily trivial, so from one 2-subgroup we find 15 new elements in addition to the 65 elements from the 5-subgroups. Since $15 + 65 = 80 = |G|$, the number of Sylow 2-subgroups cannot be more than one. Thus there exists exactly one Sylow 2-subgroup, hence it is normal.
In both cases we found a proper non-trivial normal subgroup, thus the group G cannot be simple.

4. Suppose that G is a group and $|G| = 33 = 11 \cdot 3$. Consider first the Sylow 3-subgroups of G. We have that $s_3 \equiv 1 \pmod 3$ and $s_3|11$. Thus the only possibility is that $s_3 = 1$. Denote by H the Sylow 3-subgroup of G. Since it is the only Sylow 3-subgroup, we know that it is normal.
For the Sylow 11-subgroups of G we have $s_{11} \equiv 1 \pmod{11}$ and $s_{11}|3$, and we see that also $s_{11} = 1$. Denote by K the Sylow 11-subgroup of G, which is also normal.
Then we notice that necessarily $H \cap K = \{e\}$: Since $H \cap K \leq K$, we have that $|H \cap K|$ is 1 or 11. It cannot be 11, because $H \cap K \subset H$ and $|H| = 3$, so $|H \cap K| = 1$. By Lemma 4.1 we have $HK \leq G$. Since HK has more than 11 elements, and possible orders for subgroups of G are $1, 3, 11$ and 33, we see that $HK = G$. Proposition 4.2 now gives that $G \cong H \times K \cong \mathbb{Z}_3 \times \mathbb{Z}_{11}$, which is isomorphic to $\mathbb{Z}_{33}$ by Exercise 5 a) of last week.
(The last step follows also without using the previous exercise by noticing that we have now proved that every group of order 33 is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_{11}$. Now $\mathbb{Z}_{33}$ is such a group, so they are isomorphic.)

5. a) For example

$$A^2 = B^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$BA = A^3B = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

b) The subgroup G consists of finite products of the form

$$(*) \qquad A^{k_1}B^{l_1}\cdots A^{k_n}B^{l_n},$$

where the exponents k_i, l_i are integers.
From the equation $A^4 = B^4 = I$ it follows that $A^{-1} = A^3$ and $B^{-1} = B^3$. Using this we notice that in the expression $(*)$ all negative exponents can be replaced by positive exponents.
By using the equation $BA = A^3B$ we can transform the expression to the form A^iB^j: For this first write the formula in the form where all the exponents are 1. Choose the rightmost factor B, which has one or more factors A to the right of it; then use the equation $BA = A^3B$, which allows us to "move" this factor B in such a way that there are no more factors A to the right of it. Then again choose the B which is most to the right and has an A to the right of it ... Continuing in this fashion we get to the desired form in a finite number of steps. For example

$$B^2A = BBA = BA^3B = BAAAB = A^3BAAB = A^3A^3BAB = A^3A^3A^3BB = A^9B^2.$$

From the equations $A^4 = B^4 = I$ we immediately see that with the exponents 4,5,6,... we obtain same elements as with exponents $i, j \in \{0,1,2,3\}$. In fact when we use also the equation $A^2 = B^2$ (and the equation $A^3 = AB^2$ following from this) we see that it is sufficient to consider only the exponents 0 and 1 for the element A. Thus all elements are obtained, when $i \in \{0,1\}$ and $j \in \{0,1,2,3\}$. From this we see that the group G has at most 8 elements.
c) We calculate:

$$G = \{I, A, B, AB, B^2, AB^2, B^3, AB^3\}$$

$$= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \right\}.$$

Previously we obtained that the group G has at most 8 elements. By calculating the matrices we obtained 8 different elements, so the order of G is 8.

6. Using the given assumptions we can fill part of the multiplication table:

$\cdot$	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	b	e			
b	b	e	a			
c	c			e		
d	d					
f	f					

a) Suppose that $cac^{-1} = a$, that is, $ca = ac$. From the table we notice that there are two possibilities: $ca = ac = d$ or $ca = ac = f$. Choose $ca = d$. After this choice we can fill more of the table:

$\cdot$	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	b	e	d	f	c
b	b	e	a	f	c	d
c	c	d	f	e		
d	d	f	c			
f	f	c	d			

In principal we have two possibilities: ($d^2 = f^2 = e$) or ($df = fd = e$). From the first it would follow that $e = f^2 = cbcb = cfb = cd \neq e$, which is a contradiction. Thus we must have $df = fd = e$. Moreover we have $dc = acc = ae = a$ and we can finish filling the table:

$\cdot$	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	b	e	d	f	c
b	b	e	a	f	c	d
c	c	d	f	e	a	b
d	d	f	c	a	b	e
f	f	c	d	b	e	a

By comparing this table with the table of the group $\mathbb{Z}_6$, we notice that the groups are isomorphic, the isomorphism being $e \mapsto \bar{0}$, $a \mapsto \bar{2}$, $b \mapsto \bar{4}$, $c \mapsto \bar{3}$, $d \mapsto \bar{5}$, $f \mapsto \bar{1}$.

Remark: At one step we made the choice $ca = d$. If we would have chosen $ca = f$, the table could have been filled analogously and the isomorphism would be the same as above, except that $d \mapsto \bar{1}$, $f \mapsto \bar{5}$.

b) Suppose that $cac^{-1} \neq a$, that is, $ca \neq ac$. We have two possibilities: $ca = d$ or $ca = f$. Choose $ca = d$, then we have $ac = f$, because $ca \neq ac$. We get forward:

$\cdot$	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	b	e	f	c	d
b	b	e	a	d	f	c
c	c	d	f	e		
d	d	f	c			
f	f	c	d			

Again two possibilities: $(d^2 = f^2 = e)$ or $(df = fd = e)$. From the latter it would follow that $e = df = caf = cd$, which is a contradiction. Thus $d^2 = f^2 = e$. Moreover $cd = cca = ea = a$ and we can finish filling the table:

$\cdot$	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	b	e	f	c	d
b	b	e	a	d	f	c
c	c	d	f	e	a	b
d	d	f	c	b	e	a
f	f	c	d	a	b	e

When we compare this with the table of the group S_3, see [Häsä–Rämö, p. 73], we notice that the groups are isomorphic, the isomorphism being $e \mapsto 1,\ a \mapsto \rho_1,\ b \mapsto \rho_2,\ c \mapsto \sigma_1,\ d \mapsto \sigma_2,\ f \mapsto \sigma_3$.
Similarly as in item a): If we had made the choice $ca = f$, the resulting group would be isomorphic to the group which we got.