Fourier Analysis I

Spring 2020

Homework 6

Exercise session: Thu 27 February, 14:15 - 16:00, Exactum C123; Stefanos Lappas, stefanos.lappas@helsinki.fi.

1. Let a be a real constant. Show that the equation

$$u''(x) + au'(x) - u(x) = 0 \quad \text{for all} \quad x \in \mathbb{R} \tag{1}$$

has no other 2π-periodic twice differentiable solutions than the trivial solution $u \equiv 0$.

Proof. Let u be a 2π-periodic twice differentiable function which satisfies (1). Using the Fourier series of u we can write (1) as follows

$$\sum_{n\in\mathbb{Z}} \widehat{u''}(n)e^{inx} + a\sum_{n\in\mathbb{Z}} \widehat{u'}(n)e^{inx} - \sum_{n\in\mathbb{Z}} \widehat{u}(n)e^{inx} = 0.$$

Thus,

$$\sum_{n\in\mathbb{Z}} (-n^2 + ain - 1)\widehat{u}(n)e^{inx} = 0, \tag{2}$$

where $\widehat{u'}(n) = in\widehat{u}(n)$ and $\widehat{u''}(n) = (in)^2\widehat{u}(n)$. By (2) the Fourier coefficients of u have to be 0 for all $n \in \mathbb{Z}$. Hence,

$$(-n^2 + ain - 1)\widehat{u}(n) = 0$$

and $u \equiv 0$ since a is a real constant. $\square$

2. Recall that in the lectures we described the movement of a (violin) string by the solution of the wave-equation

$$u_{tt}(x,t) = c^2 u_{xx}(x,t), \qquad u_t(x,0) = 0 \quad \text{and} \quad u(x,0) = f(x)$$

on the interval $(0,\pi)$ (with zero boundary values, initial velocity 0 and initial suspension $f(x)$) was given by the formula

$$u(x,t) = \sum_{n=1}^{\infty} c_n \cos(nct)\sin(nx) \qquad \text{with} \quad c_k = \frac{2}{\pi}\int_0^{\pi} \sin(ny)f(y)dy.$$

Try to argue that the equation is not smoothing the initial value at all as $t \to \infty$.

Proof. Let $t = \frac{2\pi}{c}$. Then,

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(nx). \tag{3}$$

Since (3) is the sine series of f and f is not smooth we conclude that the equation is not smoothing the initial value at all as $t \to \infty$. □

3. Let $f : [0, 2\pi) \to \mathbb{C}$ be a function. Given integer $N \geq 2$ define the function $F_N : Z(N) \to \mathbb{C}$ by letting

$$(*) \qquad\qquad F_N(k) := f(k2\pi/N) \qquad \text{for} \quad k = 0, 1, \dots, N-1.$$

Compute the discrete Fourier coefficients of F_N (formula (8.2) of the Finnish lecture notes) in the case

$$f(x) = e^{ax}, \qquad x \in [0, 2\pi),$$

where $a > 0$ is a constant.

Proof. We recall the formula for the discrete Fourier coefficients of F_N

$$\widehat{F_N}(n) = \frac{1}{N} \sum_{k=0}^{N-1} F_N(k) e^{-2\pi i \frac{nk}{N}}, \quad n = 0, 1, \dots, N-1.$$

Then, we compute the discrete Fourier coefficients of F_N in the case $f(x) = e^{ax}, x \in [0, 2\pi)$ as follows

$$\begin{aligned} \widehat{F_N}(n) &= \frac{1}{N} \sum_{k=0}^{N-1} e^{a \frac{2\pi k}{N}} e^{-2\pi i \frac{nk}{N}} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} e^{\frac{2\pi k}{N}(a-in)} \\ &= \frac{1}{N} \frac{e^{2\pi(a-in)} - 1}{e^{\frac{2\pi}{N}(a-ni)} - 1} \\ &= \frac{1}{N} \frac{e^{2\pi a} - 1}{e^{\frac{2\pi}{N}(a-in)} - 1}, \end{aligned}$$

where in the second equality we have a geometric sum and it holds $\frac{1}{N} \sum_{k=0}^{N-1} x^k = \frac{x^N - 1}{x-1}$, when $x = e^{\frac{2\pi}{N}(a-in)}$. □

4. Let $f \in C_\sharp[0, 2\pi)$ and for given integer $N \geq 2$ define the function F_N as in formula $(*)$ of the previous exercise. Prove that for any fixed $n \geq 1$ we obtain the (standard) Fourier coefficients of f from the discrete ones of F_N in the following way:

$$\widehat{f}(n) = \lim_{N \to \infty} \widehat{F}_N(n).$$

Check this in an example by computing the standard Fourier coefficient of the function $f(x) = e^{ax}$ (with constant $a > 0$) and comparing with the result in exercise 3.

Proof. Divide $[0, 2\pi]$ into $[\frac{2\pi k}{N}, \frac{2\pi(k+1)}{N}]$, of length $\Delta_N := \frac{2\pi}{N}$, where $k = 0, 1, \dots, N-1$. Then, recalling the definition of the Riemann sum we have

$$\begin{aligned}
\widehat{f}(n) &= \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx \\
&= \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} f\Big(\frac{2\pi k}{N}\Big) e^{-2\pi i \frac{nk}{N}} \\
&= \lim_{N \to \infty} \widehat{F_N}(n).
\end{aligned}$$

Now, we compute the Fourier coefficient of f in the following way

$$\begin{aligned}
\widehat{f}(n) &= \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx \\
&= \frac{1}{2\pi} \int_0^{2\pi} e^{ax} e^{-inx} dx \\
&= \frac{1}{2\pi} \Big(\frac{e^{(a-in)2\pi}}{a - in} - \frac{1}{a - in} \Big) \\
&= \frac{1}{2\pi} \frac{e^{(a-ni)2\pi} - 1}{a - in} \\
&= \frac{1}{2\pi} \frac{e^{2\pi a} - 1}{a - in}.
\end{aligned}$$

Recall from exercise 3 that

$$\begin{aligned}
\widehat{F_N}(n) &= \frac{1}{N} \frac{e^{2\pi a} - 1}{e^{\frac{2\pi}{N}(a-in)} - 1} \\
&= \frac{e^{2\pi a} - 1}{\frac{e^{\frac{2\pi}{N}(a-in)} - 1}{\frac{2\pi(a-in)}{N}} 2\pi(a - in)} \\
&\to \frac{1}{2\pi} \frac{e^{2\pi a} - 1}{a - in} \quad \text{as} \quad N \to \infty \\
&= \widehat{f}(n),
\end{aligned}$$

where for the limit we know that $\lim_{z\to 0}\frac{e^z-1}{z}=1$. □

5. Suppose the Fourier series of a function $g \in C_{\#}(-\pi,\pi)$ is a lacunary series of the form

$$\sum_{k=-\infty}^{\infty} a_k e^{i2^{|k|}x}.$$

Show that then the partial Fourier sums are uniformly bounded, i.e. $|S_n g(x)| \leq C$ for some constant $C < \infty$ and for all $n \in \mathbb{N}$.

Proof. Let $g \in C_{\#}(-\pi,\pi)$. Then,

$$\begin{aligned}(2F_{2^{n+1}} - F_{2^n}) * g &= 2\frac{1}{2^{n+1}}\sum_{\widetilde{n}=1}^{2^{n+1}-1} S_{\widetilde{n}}g - \frac{1}{2^n}\sum_{\widetilde{n}=1}^{2^n-1} S_{\widetilde{n}}g \\ &= \frac{1}{2^n}\sum_{\widetilde{n}=2^n}^{2^{n+1}-1} S_{\widetilde{n}}g \\ &= S_{2^n}g,\end{aligned}$$

where $S_{\widetilde{n}}g = S_{2^n}g$ for $2^n \leq \widetilde{n} \leq 2^{n+1}-1$. Hence,

$$\begin{aligned}|S_n g(x)| &= |S_{2^{n_0}}g(x)| \\ &= |(2F_{2^{n_0+1}} - F_{2^{n_0}}) * g(x)| \\ &\leq \int_{-\pi}^{\pi} |2F_{2^{n_0+1}}(x-y) - F_{2^{n_0}}(x-y)||g(y)|dy \\ &\leq \|g\|_\infty \Big(2\int_{-\pi}^{\pi} |F_{2^{n_0+1}}(x-y)|dy + \int_{-\pi}^{\pi} |F_{2^{n_0}}(x-y)|dy\Big) \\ &= \|g\|_\infty (2\|F_{2^{n_0+1}}\|_{L^1(-\pi,\pi)} + \|F_{2^{n_0}}\|_{L^1(-\pi,\pi)}) \\ &\leq \|g\|_\infty (2C + C) \\ &\leq C' < \infty,\end{aligned}$$

where in the second-to-last step we made use of the fact that Fejer kernel is a family of good kernels. □

6. Prove that the sequence $\big((2+\sqrt{3})^n\big)_{n=1}^{\infty}$ is *not* equidistributed (mod 1).

Proof. Let $x_n = (2+\sqrt{3})^n$ and $x'_n = a^n + \frac{1}{a^n}$, where $a = 2+\sqrt{3}$ and $\frac{1}{a} = 2-\sqrt{3}$. Now, recall that the sequence $(x_n)_{n=1}^{\infty}$ is equidistributed (mod 1) if the sequence of the fractional parts of $\{x_n\}$ is equidistributed in the interval $[0,1]$. Using the binomial expansion (or by induction for all $n \in \mathbb{N}$) for the sequences $(2+\sqrt{3})^n$ and $(2-\sqrt{3})^n$, we have that $x'_n \in \mathbb{N}$. Thus,

$$\{x_n\} = 1 - (2-\sqrt{3})^n \to 1 \quad \text{as} \quad n \to \infty.$$

Hence, the sequence $\left((2+\sqrt{3})^n\right)_{n=1}^{\infty}$ is *not* equidistributed (mod 1). $\square$