Algebra II

Exercise 7 (12.3.2020)

1. Apply the division algorithm to the polynomials P and S, that is, find polynomials Q, R, such that $P = QS + R$ and either R is the zero polynomial or $\deg R < \deg S$, where

a) $P = x^4 + x^3 + 2x$ and $S = x^2 - 1$

b) $P = x^3 - 3x^2 - 11x + 5$ and $S = x - 5$.

2. a) Prove that $\{(3x, y) \mid x, y \in \mathbb{Z}\}$ is a maximal ideal of the product ring $\mathbb{Z} \times \mathbb{Z}$.

b) Let R be the ring of all continuous functions $\mathbb{R} \to \mathbb{R}$. Prove that $\{f \in R \mid f(0) = 0\}$ is a maximal ideal of R.

3. Suppose that R is a (commutative) ring and A is an ideal of R. Prove that

a) A is a prime ideal, if and only if the quotient ring R/A is an integral domain.

b) A is a maximal ideal, if and only if the quotient ring R/A is a field.

4. Prove that it is not possible to define another binary operation in the group $(\mathbb{Q}/\mathbb{Z}, +)$ and obtain a ring. [Hint: Investigate the neutral element of the second binary operation. Recall that the order of every element in the group $(\mathbb{Q}/\mathbb{Z}, +)$ is finite.]

5. Suppose that R is a (commutative) ring. Prove that the ideal generated by a subset $X \subset R$ is the set

$$\langle X \rangle = \left\{ \sum_{i \in I} a_i x_i \mid I \text{ finite}, a_i \in R, x_i \in X \right\}.$$

[By definition, $\langle X \rangle$ is the smallest ideal of R containing X.]

6. a) Prove that in the ring $\mathbb{Z}[X]$ of polynomials with integer coefficients, the ideal $\langle X \rangle$ is a prime ideal, but not a maximal ideal.

b) Prove that in the ring $\mathbb{Q}[X]$ of polynomials with rational coefficients, the ideal $\langle X \rangle$ is maximal.