Algebra II. Exercise 7.
Solutions.

1. a) $x^4 + x^3 + 2x = (x^2 + x + 1) \cdot (x^2 - 1) + 3x + 1$, that is, $Q = x^2 + x + 1$ and $R = 3x + 1$.

b) $x^3 - 3x^2 - 11x + 5 = (x^2 + 2x - 1) \cdot (x - 5)$, that is, $Q = x^2 + 2x - 1$ and $R = 0$.

2. a) Denote

$$A = \{(3x, y) \mid x, y \in \mathbb{Z}\}.$$

First we prove that A is an ideal. Clearly $A \neq \emptyset$. If $(3x, y), (3x', y') \in A$, then $(3x, y) - (3x', y') = (3(x - x'), y - y') \in A$. Also, if $(r_1, r_2) \in \mathbb{Z} \times \mathbb{Z}$, $(3x, y) \in A$, then $(r_1, r_2) \cdot (3x, y) = (r_1 \cdot 3x, r_2 y) = (3 \cdot r_1 x, r_2 y) \in A$. Thus A is an ideal.

Then we prove that A is maximal. First we notice that for example $(2, 1) \notin A$, so $A \neq R$. Then assume the contrary that there exists an ideal B, such that $A \subsetneq B \subsetneq \mathbb{Z} \times \mathbb{Z}$. Then there exists an element of the form $(3x + 1, y)$ or $(3x + 2, y)$, $x, y \in \mathbb{Z}$, in the ideal B. If $(3x + 1, y) \in B$, then (because $(3x, y) \in A \subset B$) we have $(3x + 1, y) - (3x, y) = (1, 0) \in B$. If $(3x + 2, y) \in B$, then (because $(3(x + 1), y) \in A \subset B$) we have $(3(x + 1), y) - (3x + 2, y) = (1, 0) \in B$. In both cases we have $(1, 0) \in B$, and thus $(n, 0) = (n, 0) \cdot (1, 0) \in B$ for every $n \in \mathbb{Z}$. Also $(0, m) \in A \subset B$ for every m, thus

$$(n, m) = (n, 0) + (0, m) \in B \quad \text{for every } n, m \in \mathbb{Z},$$

that is, $B = \mathbb{Z} \times \mathbb{Z}$, which is a contradiction. Thus A is maximal.

b) Denote

$$A = \{f \in R \mid f(0) = 0\}$$

and denote the constant functions by $\underline{c}$, $c \in \mathbb{R}$.

First we prove that A is an ideal. Since the constant function $\underline{0} \in A$, we see that $A \neq \emptyset$. If $f_1, f_2 \in A$, then $f_1 - f_2$ is continuous and $(f_1 - f_2)(0) = f_1(0) - f_2(0) = 0 - 0 = 0$, that is, $f_1 - f_2 \in A$. If $g \in R$ and $f \in A$, then gf is continuous and $(gf)(0) = g(0) \cdot f(0) = g(0) \cdot 0 = 0$, thus $gf \in A$.

Then we prove that A is maximal. Since for example the constant function $\underline{1} \in R \setminus A$, we have $A \neq R$. Then assume the contrary that there exists an ideal B, such that $A \subsetneq B \subsetneq R$. Let $g \in B \setminus A$, then g is a continuous function and $g(0) = a \neq 0$. Now $(g - \underline{a})(0) = g(0) - \underline{a}(0) = a - a = 0$, thus $g - \underline{a} \in A \subset B$. From this it follows that $\underline{a} = g - (g - \underline{a}) \in B$, and thus also $\underline{1/a} \cdot \underline{a} = \underline{1} \in B$. Now $\underline{1}$ is the unit element of the ring R and B is an ideal, thus we have $B = R$, which is a contradiction. Thus A is maximal.

3. a) Suppose first that A is a prime ideal and we prove that R/A is an integral domain. Let $r_1 + A, r_2 + A \in R/A$ be elements, for which $(r_1 + A)(r_2 + A) = 0 + A$ (which is the zero element of the ring R/A). Then by the definition of multiplication in the quotient ring we have $r_1 r_2 + A = 0 + A$, from which it follows that $r_1 r_2 \in A$. Since A is a prime ideal, we have that $r_1 \in A$ or $r_2 \in A$, that is, $r_1 + A = 0 + A$ or $r_2 + A = 0 + A$. Thus the ring R/A doesn't have zero divisors. Furthermore, since $A \neq R$ (by the definition of a prime ideal), we have that R/A is a non-trivial ring (that is, it has at least two elements). From this it follows that R/A is an integral domain.

Then suppose that R/A is an integral domain. Suppose that $r_1, r_2 \in R$ are elements, for which $r_1 r_2 \in A$. Thus $(r_1 + A)(r_2 + A) = r_1 r_2 + A = 0 + A \in R/A$. Since R/A is an integral domain, it now follows that $r_1 + A = 0 + A$ or $r_2 + A = 0 + A$, that is, $r_1 \in A$ or $r_2 \in A$. Moreover $A \neq R$, since the ring R/A has at least two elements (by the definition of an integral domain). Thus A is a prime ideal.

b) See for example [Häsä–Rämö, Proposition 18.13, p. 227–229] or [Hungerford: Theorem 2.20, p. 128–129].

4. The elements of the group $(\mathbb{Q}/\mathbb{Z}, +)$ are cosets of the form $\frac{m}{n} + \mathbb{Z}$. Assume the contrary that we can define multiplication in such a way that $(\mathbb{Q}/\mathbb{Z}, +, \cdot)$ is a ring. Then the multiplication has a neutral element $\epsilon = \frac{m_0}{n_0} + \mathbb{Z}$ (we may assume that $n_0 \geq 1$). Now

$$n_0 \cdot \epsilon = (\frac{m_0}{n_0} + \mathbb{Z}) + ... + (\frac{m_0}{n_0} + \mathbb{Z}) = m_0 + \mathbb{Z} = \mathbb{Z} = 0_{\mathbb{Q}/\mathbb{Z}},$$

and hence for every $x \in \mathbb{Q}/\mathbb{Z}$ we have

$$n_0 \cdot x = n_0 \cdot (\epsilon x) = \epsilon x + ... + \epsilon x = (\epsilon + ... + \epsilon)x = (n_0 \cdot \epsilon) \cdot x = 0 \cdot x = 0.$$

However, if we choose $x = \frac{1}{2n_0} + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$, we get $0 = n_0 \cdot x = \frac{1}{2} + \mathbb{Z}$, that is, $\frac{1}{2} \in \mathbb{Z}$, which is a contradiction.

5. Denote

$$A = \Big\{ \sum_{i \in I} a_i x_i \mid I \text{ finite}, a_i \in R, x_i \in X \Big\}.$$

(Remark. If $X = \emptyset$, we interpret "the empty sum" as the zero element, that is, $A = \{0\}$.)
Suppose that $X \neq \emptyset$.

(I1) There exists $x \in X$, then $1 \cdot x \in A$ and thus $A \neq \emptyset$.

(I2) If $x = \sum_{i \in I} a_i x_i$, $y = \sum_{j \in J} b_j x_j$, $x, y \in A$, then $x - y$ is also a finite linear combination of elements of the set X. Thus $x - y \in A$.

(I3) If $r \in R$, then by the distributive and associative laws we have

$$r \cdot \sum_{i \in I} a_i x_i = \sum_{i \in I} r(a_i x_i) = \sum_{i \in I} (r a_i) x_i \in A.$$

Thus A is an ideal which contains the set X ($x \in X \Rightarrow x = 1 \cdot x \in A$). If B is any ideal of R containing X, then B contains all elements of the form $\sum a_i x_i$, and thus $A \subset B$. This proves that A is the smallest ideal containing X.

6. a) The ideal

$$\langle X \rangle = \{ f \cdot X \mid f \in \mathbb{Z}[X] \}$$

consists of exactly those polynomials, whose constant term is 0. Clearly $\langle X \rangle \neq \mathbb{Z}[X]$. Suppose that $fg \in \langle X \rangle$. Then the constant term of fg is 0, from which it follows that the constant term of f or the constant term of g is 0. Thus $f \in \langle X \rangle$ or $g \in \langle X \rangle$, and hence $\langle X \rangle$ is a prime ideal.
It is not a maximal ideal, since for example $\langle X \rangle \subsetneq \langle 2, X \rangle \subsetneq \mathbb{Z}[X]$. It is clear that the first inclusion is proper. Next we prove that $1 \notin \langle 2, X \rangle$, from which it follows that also the second inclusion is proper: if we had $1 \in \langle 2, X \rangle$, then by exercise 5 we would have $1 = a \cdot 2 + b \cdot X$ for some integers a, b, and clearly $b = 0$. Since there doesn't exist an integer a, for which $a \cdot 2 = 1$, the claim follows.

Alternative proof: we notice that the quotient ring $\mathbb{Z}[X]/\langle X \rangle \cong \mathbb{Z}$, which is an integral domain, but not a field. The claims now follow from Proposition 6.7 (Exercise 3).

b) If $\langle X \rangle \subsetneq A \subset \mathbb{Q}[X]$ and A is an ideal, then there exists a constant $a \in A$, $a \neq 0$. Then $\frac{1}{a} \in \mathbb{Q}$, from which it follows that $\frac{1}{a} \cdot a = 1 \in A$. Thus we have that $A = \mathbb{Q}[X]$. Hence $\langle X \rangle$ is a maximal ideal.

Alternative proof: we notice that $\mathbb{Q}[X]/\langle X \rangle \cong \mathbb{Q}$. Since $\mathbb{Q}$ is a field, the claim follows from Proposition 6.7 (Exercise 3).