Algebra II

Exercise 8 (19.3.2020)

1. a) Suppose that $m, n \in \mathbb{Z}_+$, $\gcd(m,n) = 1$. Prove that every rational number of the form $\frac{a}{mn}$ can be represented in the form $\frac{b}{m} + \frac{c}{n}$, where $b, c \in \mathbb{Z}$.
b) Prove that the above claim doesn't hold without the assumption that $\gcd(m,n) = 1$.
c) Prove that every rational number can be represented as a sum

$$\sum_{i=0}^{n} \frac{m_i}{p_i^{k_i}},$$

where the numerators are integers and the denominators are powers of prime numbers.

2. We prove the homomorphism theorem for rings. Let A and B be rings, and $f\colon A \to B$ a ring homomorphism. Prove:
a) The kernel $\mathrm{Ker}(f)$ of f is an ideal of A.
b) The image $\mathrm{Im}(f)$ is a subring of B.
c) The map f defines a (ring) isomorphism

$$\bar{f}\colon A/\mathrm{Ker}(f) \to \mathrm{Im}(f).$$

3. Suppose that $m, n \in \mathbb{Z}_+$ and denote $d = \gcd(m,n)$. Prove that $\langle m, n \rangle = \langle d \rangle$.

4. Consider the ideal generated by the polynomials $f = X^4 - 1$ and $g = X^3 + X$ in the polynomial ring $\mathbb{Z}[X]$.
a) Find a polynomial $h \in \mathbb{Z}[X]$, for which $\langle h \rangle = \langle f, g \rangle$.
b) Prove that in the quotient ring $\mathbb{Z}[X]/\langle f, g \rangle$ there exists an element a, for which $a^2 = -1$.

5. Suppose that $(M, +)$ is an Abelian group and let $n \geq 1$. Suppose that $nx = 0$ for all $x \in M$. Prove in detail that one can define a unique $\mathbb{Z}_n$-scalar multiplication $.\colon \mathbb{Z}_n \times M \to M$, such that $(M, +, .)$ is a $\mathbb{Z}_n$-module.

6. a) We prove that every vector space has a basis. Suppose that K is a field and V is a K-vector space. Consider the set $\mathcal{P}$ whose elements are all linearly independent subsets of V. The set $\mathcal{P}$ is a partially ordered set with respect to the inclusion relation. Using Zorn's lemma, prove that the set $\mathcal{P}$ has a maximal element; denote it by X. Furthermore, prove that this set X generates the space V. (Then X is a linearly independent set which generates the whole space, that is, X is a basis.)

b) Which detail in the above proof doesn't work, if we consider K-modules V, that is, K is not necessarily a field?