# Exercises 2

Please submit the exercises by e-mail at *roberto.fringuelli at helsinki.fi* by Monday, March 23, 20:00. Due to the new situation, submission by hand is no longer permitted, if you have a particular problem with this, please send me an email and we will find a solution.

There is no need to use LaTeX, you can just take a (readable) picture of your exercises and send it by email. We will discuss the exercises on Wednesday, March 25. Please write your name.

Here $k$ is an algebraically closed field.

**Exercise 1** (10 points)**.** An affine transformation $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ is a (regular) isomorphism where $f(x) = Ax + c$ where $A \in GL_n(k)$ is an invertible matrix of rank $n$ and $c \in k^n$ a vector.

(a) Show that any regular isomorphism $f : \mathbb{A}^1_k \to \mathbb{A}^1_k$ is an affine transformation.

(b) Give an example of a regular isomorphism $\mathbb{A}^m_k \to \mathbb{A}^m_k$ which is not an affine transformation, for some $m \geq 2$.

**Exercise 2** (15 points)**.** Let $X = V(x^3 - y^2) \subset \mathbb{A}^2_k$. Consider the regular morphism

$$\begin{array}{rcl} \varphi : \mathbb{A}^1_k & \longrightarrow & X \\ t & \longmapsto & (t^2, t^3) \end{array}$$

(a) Find out whether or not the restriction $\mathbb{A}^1_k \setminus \{0\} \to X \setminus V(x, y)$ is a regular isomorphism.

(b) Show that $\varphi$ is a homeomorphism (with respect to the Zariski topology).

(c) Show that $\varphi$ is not a (regular) isomorphism of affine algebraic sets.

**Exercise 3** (15 points)**.** (a) Fix $0 \neq f \in k[x_1, \ldots, x_n]$. Show that $\mathbb{A}^n_k \setminus V(f)$ is isomorphic to an affine algebraic subset of a suitable affine space $\mathbb{A}^m_k$.

(b) Let $X \subset \mathbb{A}^n_k$ be an affine algebraic set and $0 \neq f \in A(X) \cong k[x_1, \ldots, x_n]/I(X)$. Show that $X_f := X \setminus V(f)$ is isomorphic to an affine algebraic subset of a suitable affine space $\mathbb{A}^m_k$.

(c) Let $X \subset \mathbb{A}^n_k$ be a quasi-affine algebraic set. Show that there exists a (finite) cover $\{U_i\}_{i \in I}$ of open subsets in $X$ such that $U_i$ is isomorphic to an affine an affine algebraic subset of a suitable affine space $\mathbb{A}^m_k$ (Hint: use the fact that $X = V(I)/V(J)$ with $J$ finitely generated).

**Exercise 4** (10 points)**.** Let $i : \mathbb{A}^2_k \setminus V(x, y) \hookrightarrow \mathbb{A}^2_k$ be the natural inclusion.

1. Show that pull-back $i^* : k[x, y] = \mathcal{O}(\mathbb{A}^2) \to \mathcal{O}(\mathbb{A}^2_k \setminus V(x, y))$ of rings of regular functions is an isomorphism of $k$-algebras (Hint: study the restriction of a regular function of $\mathbb{A}^2_k \setminus V(x, y)$ to the open cover $\{\mathbb{A}^2_k \setminus V(x), \mathbb{A}^2_k \setminus V(y)\}$).
2. Deduce that $\mathbb{A}^2 \setminus \{0, 0\}$ is not an affine variety, i.e. it is not isomorphic to an irreducible closed subset of a suitable affine space $\mathbb{A}^m_k$.