Exercises 1

The exercises are not necessary for passing the course. However, they will provide additional points for the exam.
Please submit the exercises by e-mail by Monday, March 16, 20:00 or by hand during the lecture of Monday. We will discuss them on Wednesday, March 18. Please write your name.

We always assume that k is an algebraically closed field.

Exercise 1 (10 points)**.** Let R be a ring.

(a) Assume R Noetherian. Show that R/I is Noetherian for any ideal $I \subset R$ (*Hint:* Describe the ideals in R/I in terms of certain ideals in R).

(b) Find an example of a ring R which is not Noetherian.

(c) Let I and J be ideals in R. Show that $IJ := \{\sum_i f_i g_i \mid f_i \in I \text{ and } g_i \in J\} \subset I \cap J$. Give an example of ideals I and J such that $IJ \neq I \cap J$.

(d) Let I and J be ideals in R. Show that $V(IJ) = V(I) \cup V(J)$.

Exercise 2 (10 Points)**.** Prove that an algebraic set in $\mathbb{A}^1_k$ is either finite or the entire affine space $\mathbb{A}^1_k$.

Exercise 3 (10 points)**.** Which one of the following subsets of $\mathbb{A}^2_{\mathbb{C}} := \mathbb{C}^2$ is a quasi-affine algebraic set (i.e. locally closed subset for the Zariski topology)? Which one is affine (i.e. closed subset for the Zariski topology)?

(a) $Z_0 := \{(x, y) \in \mathbb{A}^2_{\mathbb{C}} \mid |x| + |y| = 1\}$, where $|a + ib| := a^2 + b^2$ for any a, b in $\mathbb{R}$.

(b) $Z_1 := \{(t, t^2) \in \mathbb{A}^2_{\mathbb{C}} \mid t \in \mathbb{C}\}$.

(c) $Z_2 := \left\{\left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}\right) \in \mathbb{A}^2_{\mathbb{C}} \mid t \in \mathbb{C} \setminus \{\pm i\}\right\}$.

(d) $Z_3 := \{(t, tu) \in \mathbb{A}^2_{\mathbb{C}} \mid (t, u) \in \mathbb{A}^2_{\mathbb{C}}\}$.

Exercise 4 (10 points)**.** Compute $I(X)$ for

(a) $X = \mathbb{Z}^2 \subset \mathbb{A}^2_{\mathbb{C}}$,

(b) $X = V(x^2 + y^2 - 1, y - 1) \subset \mathbb{A}^2_{\mathbb{C}}$. Show that $(x^2 + y^2 - 1, y - 1) \neq I(X)$.

(c) $X = V(x^2 + y^2 + z^2) \in \mathbb{A}^2_k$, if the characteristic of the field is 2.

(d) $X = V(x^2 + y^2 + z^2) \in \mathbb{A}^2_k$, if the characteristic of the field is not 2.

Exercise 5 (10 points)**.** Describe the irreducible components X of $V(I)$ and determine their ideals $I(X)$ for

(a) $I = (x^2 + x + 1) \subset \mathbb{C}[x]$,

(b) $I = (xy, xz, yz) \in \mathbb{C}[x, y]$,

(c) $I = (y^2 - xy - x^2y + x^3) \in \mathbb{C}[x, y]$,

(d) $I = (x^2 - yz, xz - x) \in \mathbb{C}[x, y, z]$.