FOURIER ANALYSIS II. (spring)

1. EXERCISES (Monday 23.3)

NOTE: return these exercises at the latest as pdf and by e-mail (clear hand-writing, or typed) to

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They should be returned at the latest during the day os the 'exercise class'. The course will be evaluated based on the returned exercises.

1. Compute the Fourier transform of the characteristic function $\chi_{[-a,a]}$ (here $a > 0$).

2. **(i)** Compute the convolution $\chi_{[-a,a]} * \chi_{[-a,a]}$.

(ii) Compute the Fourier transform (in one dimension) of the function $g(x) = \max(0, 1 - |x|)$.

3. Compute the Fourier transform $f : \mathbf{R} \to \mathbf{R}$, where $f(x) := e^{-k|x|}$ (here$k > 0$) : show that

$$\widehat{f}(\xi) = \frac{2k}{k^2 + \xi^2}$$

4. **(i)** If $f \in L^1(\mathbf{R}^d)$ and $g(x) = \overline{f(-x)}$, show that $\widehat{g}(\xi) \equiv \overline{\widehat{f}(\xi)}$.

(ii) If $f \in L^1(\mathbf{R}^d)$ and $g(x) = \frac{1}{t^d} f\left(\frac{x}{t}\right)$, $t > 0$, show that $\widehat{g}(\xi) \equiv \widehat{f}(t\,\xi)$.

5. Suppose that the function $f : \mathbf{R}^d \to \mathbf{C}$ has the form

$$f(x_1, x_2) = f_1(x_1) f_2(x_2) \cdots f(x_d), \qquad \forall\, x = (x_1, \ldots, x_d) \in \mathbf{R}^d,$$

where $f_1, \ldots, f_d \in L^1(\mathbf{R}^1)$. Show that then $f \in L^1(\mathbf{R}^d)$ and we have

$$\widehat{f}(\xi) = \widehat{f}_1(\xi_1) \widehat{f}_2(\xi_2) \cdots \widehat{f}_d(\xi_d) \qquad \forall\, \xi = (\xi_1, \ldots, \xi_d) \in \mathbf{R}^d.$$

6. Assume that $H \in L^1(\mathbf{R}^d)$ fulfils $H \geq 0$, and $\int_{\mathbf{R}^d} H(x)dx = 1$, together with

$$|H(x)| \leq \frac{C}{(1 + |x|)^{d+1}}.$$

For $\varepsilon > 0$ let us denote $H_\varepsilon(x) := \varepsilon^{-d} H(x/\varepsilon)$. If $f \in L^1(\mathbf{R}^d)$ is continuous at 0, prove that

$$\lim_{\varepsilon \to 0} \int_{\mathbf{R}^d} f(x) H_\varepsilon(x) dx \;=\; f(0).$$

7. Let $a > 0$. Check that the function $H(x) := c_a e^{-a|x|^2}$ with suitable constant c_a satisfies the conditions of the previous exercise. What is the value of c_a?

8^{*1} Prove Leibnitz general rule for differentiation of products: if $\alpha \in \mathbf{N}_0^d$ is an arbitrary multi-index and $f, g \in C^\infty(\mathbf{R}^d)$, then

$$\partial^\alpha (fg)(x) \;=\; \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta f(x)\, \partial^{\alpha-\beta} g(x),$$

where $\displaystyle \binom{\alpha}{\beta} \;:=\; \prod_{j=1}^{d} \binom{\alpha_j}{\beta_j}$

[1]These $*$-exercises are extras for afficinadoes, not required to get full points from exercises