Algebra II

Exercise 9 (26.3.2020)

1. Suppose that R is a (commutative) ring. Consider the ideal $\langle X\rangle$ of the polynomial ring $R[X]$. Prove (for example by using the homomorphism theorem for rings) that

$$R[X]/\langle X\rangle \cong R.$$

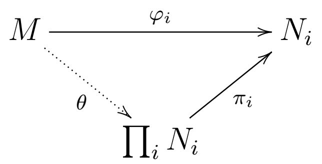
2. Prove that every linearly independent subset of the $\mathbb{Z}$-module $\mathbb{Q}$ contains at most one element, and deduce from this that $\mathbb{Q}$ is not a free group. (An Abelian group is called *free*, if it is free as a $\mathbb{Z}$-module.)

3. Suppose that R is a ring with an ideal A and subring S. Prove:

a) Every R/A-module is also an R-module, but every R-module is not an R/A-module. [Hint: $\mathbb{Z}, \mathbb{Z}_n, \mathbb{Q}$]

b) Every R-module is also an S-module, but every S-module is not an R-module. [Hint: $\mathbb{Z}, \mathbb{Q}$]

4. Prove the universal property of the direct product of modules and the canonical projections. Let (N_i) be a family of R-modules. Denote the canonical projections by π_i. Suppose also that M is an R-module, and for every index i we are given an R-linear map $\varphi_i\colon M \to N_i$. Then there exists a unique R-linear map $\theta\colon M \to \prod_i N_i$, for which $\varphi_i = \pi_i \circ \theta$ for every i, that is, the following diagram commutes:



5. Suppose that $f\colon A \to B$ and $g\colon B \to A$ are homomorphisms between R-modules, for which $g \circ f = \mathrm{id}_A$. Prove that

$$B \cong \mathrm{Im}(f) \oplus \mathrm{Ker}(g).$$

6. Prove by giving a counterexample that the direct sum (and the projection maps associated with it) doesn't in general have the universal property of Exercise 4.