FOURIER ANALYSIS II. (spring 2020)

1. EXERCISES (Thursday 26.3)

NOTE: return these exercises at the latest as pdf and by e-mail (clear hand-writing, or typed) to

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They should be returned at the latest during the day of the 'exercise class'. The course will be evaluated based on the returned exercises.

1-2. (a specially important exercise!) Let $\alpha \in \mathbf{N}_0^d$ be a multi-idex. Prove with all details that if $f \in \mathcal{S}(\mathbf{R}^d)$, then

(i) $x^\alpha f(x) \in \mathcal{S}(\mathbf{R}^d)$ and $\partial^\alpha f(x) \in \mathcal{S}(\mathbf{R}^d)$,

(ii) $\widehat{f} \in C^\infty(\mathbf{R}^d)$.

(iii) $(\partial^\alpha f)\widehat{\ }(\xi) = (i\,\xi)^\alpha \widehat{f}(\xi)$ (note that one defines $i^\alpha := i^{|\alpha|}$).

(iv) Apply part (iii) by choosing suitable multi-indices α to verify that $\widehat{f}$ decays any polynomial rate, i.e. for any $N \geq 1$ there is a constant C so that $|\widehat{f}(\xi)| \leq C(1+|\xi|^2)^{-N}$.

3. (a specially important exercise!) Apply the previous exercise and verify carefully that

$$\text{if} \quad f \in \mathcal{S}(\mathbf{R}^d), \quad \text{then} \quad \widehat{f} \in \mathcal{S}(\mathbf{R}^d).$$

4. Which of the following functions belong to $\mathcal{S}(\mathbf{R}^d)$?

(i) $f(x) = (1+|x|^2)^{-1}$. **(iii)** $f(x) = e^{-|x|^2}$.

(i) $f(x) = e^{-|x|^2}\cos(e^{|x|^2})$.

5. Compute the integral $\displaystyle\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^2 dx$ by first computing the Fourier transform of the characteristic function $\chi_{[-1,1]}$.

[Hint: what happens with the L^2-norms...?]

6. Assume that $f \in \mathcal{S}(\mathbf{R}^d)$. **(i)** Compute the Fourier transform of the Laplacian $\Delta f := \big(\sum_{j=1}^d \big(\frac{\partial}{\partial x_j}\big)^2\big) f$ in terms of $\widehat{f}$.

(ii) Show that $\dfrac{f(x)}{1+|x|^2} \in \mathcal{S}(\mathbf{R}^d)$.

7. Use Fourier transform to find a solution formula for the partial differential equation

$$\Delta f - f = g$$

for given $g \in \mathcal{S}(\mathbf{R}^d)$ and show that also the solution f lies in $\mathcal{S}(\mathbf{R}^d)$.

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(ii) Specialize in the previous exercise to dimension $d = 1$ and show that the solution is given by the convolution

$$f(x) = -\frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|}g(y)dy.$$

(ii) Given $\varepsilon > 0$, show that one may pick $g \in \mathcal{S}(\mathbf{R}^d)$ so that the solution f satisfies $\|f\|_{L^2(\mathbf{R})} < \varepsilon \|g\|_{L^2(\mathbf{R})}$.

[1] These $*$-exercises are extras for afficinadoes, not required to get full points from exercises