Complex Analysis Homework 1 Solutions

1.1

Show that

$$\overline{zw} = \bar{z}\bar{w} \text{ for all } z, w \in \mathbb{C}.$$

With a direct calculation we get

$$\begin{aligned}
\overline{zw} &= \overline{(z_1 + iz_2)(w_1 + iw_2)} \\
&= \overline{z_1 w_1 - z_2 w_2 + i(z_1 w_2 + z_2 w_1)} \\
&= z_1 w_1 - z_2 w_2 - i(z_1 w_2 + z_2 w_1) \\
&= z_1 w_1 - i(z_1 w_2 + z_2 w_1) + i^2 z_2 w_2 \\
&= (z_1 - iz_2)(w_1 - iw_2) \\
&= \bar{z}\bar{w}
\end{aligned}$$

1.2

Find the real and imaginary parts of the complex numbers

$$\frac{4}{(3-i)^2}, \quad \frac{1-2i}{(3+4i)^3}, \quad \left(\frac{1-i}{i+1}\right)^{11}$$

With few steps we can express the complex numbers in more familiar form $z = x + iy$.

$$z = \frac{4}{(3-i)^2} = \frac{4}{8-6i}\frac{8+6i}{8+6i} = \frac{4(8+6i)}{100} = \frac{8}{25} + \frac{6}{25}i$$

Hence we have $\text{Re}(z) = \frac{8}{25}$ and $\text{Im}(z) = \frac{6}{25}$.

$$\begin{aligned}
w &= \frac{1-2i}{(3+4i)^3} = \frac{1-2i}{27 + 3\cdot 9(4i) + 3\cdot 3\cdot(-16) - 64i} = \frac{1-2i}{44i - 117}\frac{44i+117}{44i+117} \\
&= \frac{205 - 190i}{-15625} = -\frac{41}{3125} + \frac{38}{3125}i
\end{aligned}$$

Hence we have $\text{Re}(w) = -\frac{41}{3125}$ and $\text{Im}(w) = \frac{38}{3125}$.

$$
\begin{aligned}
u &= \left(\frac{i-1}{i+1}\right)^{11} = \left(\frac{i-1}{i+1}\frac{1-i}{1-i}\right)^{11} = \left(\frac{2i}{2}\right)^{11} \\
&= (i)^{11} = i(i^2)^5 = i(-1)^5 = -i
\end{aligned}
$$

Hence we have $\mathrm{Re}(u) = 0$ and $\mathrm{Im}(u) = -1$.

1.3

Let $z \in \mathbb{C} \setminus \{0\}$**. Show that**

$$
z + \frac{1}{z} \in \mathbb{R} \textbf{ if and only if Im } z = 0 \textbf{ or } |z| = 1.
$$

$(\implies)$ Let $z + \frac{1}{z} \in \mathbb{R}$, that is $\mathrm{Im}(z + \frac{1}{z}) = 0$. Since

$$
z + \frac{1}{z} = z + \frac{\bar{z}}{|z|^2} = \frac{z|z|^2 + \bar{z}}{|z|^2},
$$

then

$$
\mathrm{Im}(\frac{z|z|^2 + \bar{z}}{|z|^2}) = 0,
$$

which means $\mathrm{Im}(z|z|^2 + \bar{z}) = 0$. Now

$$
\mathrm{Im}(z|z|^2 + \bar{z}) = \mathrm{Im}(z|z|^2) + \mathrm{Im}(\bar{z}) = \mathrm{Im}(z|z|^2) - \mathrm{Im}(z) = \mathrm{Im}(z)(|z|^2 - 1),
$$

so either $\mathrm{Im}(z) = 0$ or $|z|^2 = 1$.

$(\impliedby)$ Im $z = 0$ or $|z| = 1$

$$
z + \frac{1}{z} = z + \frac{\bar{z}}{z\bar{z}} = z + \frac{\bar{z}}{|z|^2} = z + \bar{z} \in \mathbb{R}
$$

hence the equivalence is shown.

Alternatively we can expand $z = x + iy$ and we get the same result from the following.
$(\implies)$

$$\begin{aligned} z + \frac{1}{z} &= x + iy + \frac{1}{x+iy}\frac{x-iy}{x-iy} \\ &= \frac{\big(x+iy\big)\big(x^2+y^2\big) + x - iy}{x^2+y^2} \\ &= \frac{x^3 + xy^2 + x + iy^3 - iy}{x^2+y^2} \\ &= \frac{x\big(x^2+y^2\big) + x + i\big(y(x^2+y^2) - y\big)}{x^2+y^2} \\ &= x + \frac{x}{x^2+y^2} + i\Big(y - \frac{y}{x^2+y^2}\Big) \\ &= x + \frac{x}{x^2+y^2} + i\Big(y\big(1 - \frac{1}{x^2+y^2}\big)\Big) \end{aligned}$$

Since $z + \frac{1}{z} \in \mathbb{R}$ we know that $y\big(1 - \frac{1}{x^2+y^2}\big) = 0$ which gives us $y = 0$ or $x^2 + y^2 = 1$, that is, $\operatorname{Im} z = 0$ or $|z| = 1$.

$(\impliedby)$ Clearly if $\operatorname{Im}(z) = 0$ the sum $z + \frac{1}{z}$ is as sum of two real numbers a real number. Now let $|z| = 1 \implies x^2 + y^2 = 1$

$$\begin{aligned} z + \frac{1}{z} &= x + \frac{x}{x^2+y^2} + i\Big(y\big(1 - \frac{1}{x^2+y^2}\big)\Big) \\ &= x + \frac{x}{x^2+y^2} \in \mathbb{R}. \end{aligned}$$

1.4

(De Moivre's Theorem). **Let** $\alpha \in \mathbb{R}$**. Prove that**

$$(\cos\alpha + i\sin\alpha)^n = \cos(n\alpha) + i\sin(n\alpha)$$

for all $n \in \mathbb{Z}$**.**

Clearly the formula holds for $n = 0$. Now for the proof by induction we have

$$\begin{aligned}
&\Big(\cos(\alpha) + i\sin(\alpha)\Big)^{n+1} \\
&= \Big(\cos(\alpha) + i\sin(\alpha)\Big)^n \Big(\cos(\alpha) + i\sin(\alpha)\Big) \\
&= \Big(\cos(n\alpha) + i\sin(n\alpha)\Big)\Big(\cos(\alpha) + i\sin(\alpha)\Big) \\
&= \cos(n\alpha)\cos(\alpha) - \sin(n\alpha)\sin(\alpha) + i\Big(\cos(n\alpha)\sin(\alpha) + \cos(\alpha)\sin(n\alpha)\Big) \\
&= \cos\big((n+1)\alpha\big) + i\sin\big((n+1)\alpha\big)
\end{aligned}$$

Hence the formula holds for all $n \in \mathbb{Z}_+$. Now to prove that it also holds for negative integers. Suppose $n \in \mathbb{Z}_+$

$$\begin{aligned}
\Big(\cos(\alpha) + i\sin(\alpha)\Big)^{-n} &= \Big(\cos(n\alpha) + i\sin(n\alpha)\Big)^{-1} \\
&= \frac{1}{\cos(n\alpha) + i\sin(n\alpha)} \frac{\cos(n\alpha) - i\sin(n\alpha)}{\cos(n\alpha) - i\sin(n\alpha)} \\
&= \frac{\cos(n\alpha) - i\sin(n\alpha)}{\cos^2(n\alpha) + \sin^2(n\alpha)} \\
&= \cos(n\alpha) - i\sin(n\alpha) \\
&= \cos(-n\alpha) + i\sin(-n\alpha)
\end{aligned}$$

□

1.5

Let $z = \sqrt{3} - i$ Find the magnitude of z and the arguments $\mathbf{Arg}_{(-\pi,\pi]}$ and $\mathbf{Arg}_{(0,2\pi]}$ of z. Find the modulus and argument $\mathbf{Arg}_{(0,2\pi]}$ of the number $\left(\sqrt{3} - i\right)^5$

Let's start by calculating magnitude and angle of $z = \sqrt{3} - i$.

$$|z| = \sqrt{3+1} = 2$$

$$\tan\alpha = \frac{-1}{\sqrt{3}} \implies \alpha = \tan^{-1}(\frac{-1}{\sqrt{3}}) = -\frac{\pi}{6}$$

So for z we have the following

$$\text{Arg}_{(-\pi,\pi]}(z) = -\frac{\pi}{6}$$

and

$$\text{Arg}_{(0,2\pi]}(z) = \frac{11}{6}\pi$$

For $z = \left(\sqrt{3} - i\right)^5$ using the answer from the previous part and De Moivre's theorem we have

$$z = \left(\sqrt{3} - i\right)^5 = \left(2\cos(\frac{11}{6}\pi) + 2i\sin(\frac{11}{6}\pi)\right)^5 = 32\Big(\cos(\frac{55}{6}\pi) + i\sin(\frac{55}{6}\pi)\Big)$$

with this we can easily see that

$$|z| = 32$$

and

$$\text{Arg}_{(0,2\pi]}(z) = \frac{7}{6}\pi.$$

1.6 (An optional extra problem.)

The mapping $z \mapsto \bar{z}$ is a reflection across the real axis in the plane. Let L be a line passing through the origin and forming an angle α with counterclockwise orientation between L and the positive real axis. Find a mapping which gives a reflection across L

Since L goes through the point $(0,0)$ the magnitude of the reflection does not change. Reflecting z across the line L should only flip the sign of the angle between L and the polar representation of z, this gives us

$$\alpha - \mathrm{Arg}_{(0,2\pi]}(z) = -\big(\alpha - \mathrm{Arg}_{(0,2\pi]}(z_{\text{reflection}})\big)$$
$$\mathrm{Arg}_{(0,2\pi]}(z_{\text{reflection}}) = 2\alpha - \mathrm{Arg}_{(0,2\pi]}(z)$$

where α is the angle between L and the positive real axis. With these the wanted mapping is $f : \mathbb{C} \to \mathbb{C}, \quad z \mapsto |z|e^{i\left(2\alpha - \mathrm{Arg}_{(0,2\pi]}(z)\right)}$