Complex Analysis Homework 2
Solutions

2.1

(1) Express $\cos 5\alpha$ by using a linear combination of terms $\cos^k \alpha, k \geq 1$.
(2) Solve the equation $z^6 + 1 = \sqrt{3}i$. Draw a picture which shows the solutions in the complex plane.

(1) With a direct calculation using De Moivre's formula we get

$$
\begin{aligned}
\cos(5\alpha) &= \frac{1}{2} 2\cos(5\alpha) \\
&= \frac{1}{2}\Big(\cos(5\alpha) + i\sin(5\alpha) + \cos(5\alpha) - i\sin(5\alpha)\Big) \\
&= \frac{1}{2}\Big(\big(\cos\alpha + i\sin\alpha\big)^5 + \big(\cos\alpha - i\sin\alpha\big)^5\Big) \\
&= \frac{1}{2}\Big(\cos^5\alpha - 10\cos^3\alpha\sin^2\alpha + 5\cos\alpha\sin^4\alpha\Big) \\
&= \frac{1}{2}\Big(\cos^5\alpha - 10\cos^3\alpha\big(1-\cos^2\alpha\big)\alpha + 5\cos\alpha\big(1-\cos^2\alpha\big)^2\Big) \\
&= 16\cos^5\alpha - 20\cos^3\alpha + 5\cos\alpha
\end{aligned}
$$

(2) The distinct roots of $z^6 = \sqrt{3}i - 1 = a$ are given by

$$z_k = |a|^{\frac{1}{n}} e^{\frac{\theta + k2\pi}{n}}, \quad k = 0, \ldots, n-1$$

Let's start by expressing a in a polar form.

$$|a| = \big((-1)^2 + 3\big)^{\frac{1}{2}} = 2$$

$$2\cos\theta = -1 \implies \theta = \frac{2}{3}\pi$$

With these we get $a = 2e^{\frac{2}{3}\pi i}$ and the roots are given by

$$z_k = 2^{\frac{1}{6}} e^{(\frac{1}{9}\pi + \frac{k\pi}{3})i}$$

$$z_0 = 2^{\frac{1}{6}} e^{\frac{1}{9}\pi i}, z_1 = 2^{\frac{1}{6}} e^{\frac{4}{9}\pi i}, z_2 = 2^{\frac{1}{6}} e^{\frac{7}{9}\pi i}, z_3 = 2^{\frac{1}{6}} e^{\frac{10}{9}\pi i}, z_4 = 2^{\frac{1}{6}} e^{\frac{13}{9}\pi i}, z_5 = 2^{\frac{1}{6}} e^{\frac{16}{9}\pi i}$$

2.2

Let $f : \mathbb{C} \setminus \{-i, i\} \mapsto \mathbb{C}$ be a function defined by

$$f(z) = \Big(\frac{z^2 - 1}{z^2 + 1}\Big)^9.$$

When is the function f analytic? Calculate the derivative $f'(z)$.

The given function is a composition $f(g(z))$ where

$$f(z) = z^9$$
$$g(z) = \frac{z^2 - 1}{z^2 + 1}.$$

We can see that g is a rational function $\frac{p(z)}{q(z)}$ and as such is analytic when $q(z) \neq 0$. For this we need to find the roots of $z^2 + 1$. Which are given by

$$z_k = e^{\frac{\pi + k2\pi}{2} i}$$

which gives us the roots

$$z_1 = e^{\frac{1}{2}\pi i} = i \text{ and } z_2 = e^{\frac{3}{2}\pi i} = -i.$$

The roots are not in the domain of f and hence the function is analytic in its domain since composite function of analytic functions is analytic. As a composite function the derivative of f is

$$f(g(z)) = f'(g(z))g'(z)$$

where $g'(z)$ is

$$\frac{d}{dz}\frac{z^2 - 1}{z^2 + 1} = \frac{2z(z^2 + 1) - 2z(z^2 - 1)}{(z^2 + 1)^2} = \frac{4z}{(z^2 + 1)^2}.$$

Now we have

$$f'(g(z)) = 9\Big(\frac{z^2 - 1}{z^2 + 1}\Big)\frac{4z}{(z^2 + 1)^2} = \frac{36z(z^2 - 1)^8}{(z^2 + 1)^{10}}$$

2.3

(1)**When is the function**

$$f(z) = \frac{1}{(z+1/z)^3}$$

analytic? Find the derivative of f.
(2) **Consider the function defined by**

$$f(x+iy) = \sqrt{|x||y|}, \text{ **, whenever** } x, y \in \mathbb{R}$$

Is the function f complex differentiable at the origin?

(1) The given function is a composite function $f = (g \circ h)(z)$ where

$$g(z) = \frac{1}{z^3}$$
$$h(z) = z + \frac{1}{z}.$$

From these we can see that f is analytic when $z \neq 0$ and $z + 1/z \neq 0$

$$z = -\frac{1}{z}$$
$$z^2 = -1$$
$$z = \pm i$$

so f is analytic in $\mathbb{C} \setminus \{0, i, -i\}$.

Now to calculate the derivative of f.

$$(g \circ h)'(z) = g'(h(z))h'(z)$$

$$= g'(z) = \frac{-3}{z^4}$$
$$= h'(z) = 1 - \frac{1}{z^2}$$

with these we have

$$(g \circ h)'(z) = \frac{-3}{\left(z + \frac{1}{z}\right)^4}\left(1 - \frac{1}{z^2}\right)$$
$$= -3\frac{\left(\frac{z^2-1}{z^2}\right)}{\left(\frac{z^2+1}{z}\right)^4}$$
$$= -3\frac{z^4(z^2-1)}{z^2(z^2+1)^4}$$
$$= -\frac{3z^2(z^2-1)}{(z^2+1)^4}$$

(2) Consider the derivative along the path $z(t) = t + it$, now

$$\begin{aligned}\lim_{t \to 0} \frac{f(t+it) - f(0)}{t + it} &= \lim_{t \to 0} \frac{\sqrt{|t||t|}}{t + it} \\ &= \lim_{t \to 0} \frac{|t|}{t + it} \\ &= \lim_{t \to 0} \frac{|t|}{t(1+i)}\end{aligned}$$

here we can see two separate situations, namely if $t < 0$ or if $t \geq 0$. Firstly suppose t is greater than zero, now

$$\lim_{t \to 0} \frac{|t|}{t(1+i)} = \lim_{t \to 0} \frac{t}{t(1+i)} = \frac{1-i}{2}$$

but when $t < 0$ we have

$$\lim_{t \to 0} \frac{|t|}{t(1+i)} = \lim_{t \to 0} \frac{-t}{t(1+i)} = \frac{i-1}{2}$$

therefore f is not complex differentiable at the origin.

2.4

Let $x \in \mathbb{R}$. Consider complex valued functions f and g defined in the disc $\mathbb{D}(x, 10)$ by

$$g(z) = \overline{f(\bar{z})}.$$

Prove that g is analytic in $\mathbb{D}(x, 10)$ if and only if f is analytic in $\mathbb{D}(x, 10)$.

Let $z \in \mathbb{D}(x, 10)$ and notice that $g(z) = \overline{f(\bar{z})} \implies \overline{g(z)} = f(\bar{z})$.

($\implies$) Suppose that g is analytic and take a look at the limit

$$\begin{aligned}f'(\bar{z}) &= \lim_{h \to 0} \frac{f(\overline{z+h}) - f(\bar{z})}{\bar{h}} \\ &= \lim_{h \to 0} \frac{\overline{g(z+h) - g(z)}}{\bar{h}} \\ &= \lim_{h \to 0} \overline{\left(\frac{g(z+h) - g(z)}{h}\right)} \\ &= \overline{\left(\lim_{h \to 0} \frac{g(z+h) - g(z)}{h}\right)} = \overline{g'(z)}.\end{aligned}$$

Since g is analytic in $\mathbb{D}(x, 10)$ the limit exists if $\bar{z} \in \mathbb{D}(x, 10)$. For this remember that $x \in \mathbb{R}$, with this we can see that

$$\begin{aligned}
|x - z| &= |x_1 + ix_2 - z_1 - iz_2| \\
&= |x_1 - z_1 - iz_2| \\
&= \sqrt{(x_1 - z_1)^2 + (-z_2)^2} \\
&= \sqrt{(x_1 - z_1)^2 + (z_2)^2} \\
&= |x_1 - z_1 + iz_2| \\
&= |x - \bar{z}| \leq 10
\end{aligned}$$

hence $\bar{z} \in \mathbb{D}(x, 10)$ the openness of the disc also guarantees that $\bar{z} + h \in \mathbb{D}(x, 10)$ as $h \to 0$ and therefore the limit exists and f is analytic in $\mathbb{D}(x, 10)$.

($\Longleftarrow$) Suppose that f is analytic.

Since f is analytic in $\mathbb{D}(x, 10)$ we know that $f'(z)$ exists and therefore the uniqueness of the limit gives us

$$f'(\bar{z}) = \overline{g'(z)} \implies g'(z) = \overline{f'(\bar{z})}$$

and hence the function g is analytic in $\mathbb{D}(x, 10)$. $\square$

2.5

Find the set where the function

$$f(z) = \frac{z}{1+|z|}$$

is complex differentiable.

First suppose that $z = 0$ then:

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{\frac{h}{1+|h|}}{h} = \lim_{h \to 0} \frac{1}{1+|h|} = 1,$$

so f is a complex differentiable at $z = 0$.

Now suppose that f is complex differentiable at $z \neq 0$, then

$$\frac{z}{f(z)}$$

would be a division of two complex differentiable functions and therefore complex differentiable. Which in turn implies that

$$\frac{z}{f(z)} = \frac{z}{z}(1+|z|) = 1+|z|$$

is complex differentiable. But this is a contradiction since $|z|$ is not complex differentiable anywhere[1]. Hence the function f is only differentiable in the singelton set $\{0\}$.

[1] *Proof.*
Suppose that $|z|$ is complex differentiable, then $\frac{|z|^2}{z} = \frac{z\bar{z}}{z} = \bar{z}$ would also be, but this is a contradiction since $\bar{z}$ is not complex differentiable anywhere.

□