Complex Analysis Homework 3 Solutions

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Exercise 1. Suppose that $f\colon \mathbb{C} \to \mathbb{C}$ is entire. Prove that the function $g(z) = \bar{z}f(z)$ is differentiable at the point z if and only if $f(z) = 0$.

Solution

Method 1, using the Cauchy-Riemann equations. Let $z = x + iy \in \mathbb{C}$. Let $f(z) = u(x, y) + iv(x, y)$, where $u\colon \mathbb{R}^2 \to \mathbb{R}$ and $v\colon \mathbb{R}^2 \to \mathbb{R}$.

Obtain the following expression for g

$$\begin{aligned} g(z) &= \bar{z}f(z) \\ &= \bar{z}\left(u(x, y) + iv(x, y)\right) \\ &= (x - iy)\left(u(x, y) + iv(x, y)\right) \\ &= xu(x, y) + yv(x, y) + i\left(xv(x, y) - yu(x, y)\right). \end{aligned}$$

Denote

$$\begin{aligned} s&\colon \mathbb{R}^2 \to \mathbb{R}, \quad (x, y) \mapsto xu(x, y) + yv(x, y) \\ t&\colon \mathbb{R}^2 \to \mathbb{R}, \quad (x, y) \mapsto xv(x, y) - yu(x, y). \end{aligned}$$

Now $g\colon \mathbb{C} \to \mathbb{C}$, where $g(z) = s(x, y) + it(x, y)$.

The partial derivatives of s and t

$$\begin{aligned} \frac{\partial s}{\partial x} &= u(x, y) + x\frac{\partial u}{\partial x} + y\frac{\partial v}{\partial x} \\ \frac{\partial t}{\partial x} &= v(x, y) + x\frac{\partial v}{\partial x} - y\frac{\partial u}{\partial x} \\ \frac{\partial s}{\partial y} &= x\frac{\partial u}{\partial y} + v(x, y) + y\frac{\partial v}{\partial y} \\ \frac{\partial t}{\partial y} &= x\frac{\partial v}{\partial y} - u(x, y) - y\frac{\partial u}{\partial y}. \end{aligned}$$

Assume that g is differentiable at z. According to corollary 4.3.11 the Cauchy-Riemann equations hold for g at z, that is

$$\frac{\partial s}{\partial x} = \frac{\partial t}{\partial y} \quad \text{and} \quad \frac{\partial t}{\partial x} = -\frac{\partial s}{\partial y}.$$

Replacing the partial derivatives with the expressions calculated above

$$\begin{aligned} u(x,y) + x\frac{\partial u}{\partial x} + y\frac{\partial v}{\partial x} &= x\frac{\partial v}{\partial y} - u(x,y) - y\frac{\partial u}{\partial y}, \text{ and} \\ v(x,y) + x\frac{\partial v}{\partial x} - y\frac{\partial u}{\partial x} &= -x\frac{\partial u}{\partial y} - v(x,y) - y\frac{\partial v}{\partial y} \end{aligned} \tag{1}$$

Since function f is entire, that is f is analytic on $\mathbb{C}$, the Cauchy-Riemann equations hold for f:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Consequently, the equations (1) are now

$$\begin{aligned} u(x,y) &= -u(x,y) \\ v(x,y) &= -v(x,y). \end{aligned}$$

Since $u(x,y) \in \mathbb{R}$ and $v(x,y) \in \mathbb{R}$, the Cauchy-Riemann equations hold for g if $u(x,y) = v(x,y) = 0$. Thus $f(z) = u(x,y) + iv(x,y) = 0$.

Assume now that $f(z) = u(x,y) + iv(x,y) = 0$, so $u(x,y) = v(x,y) = 0$. From previous calculations obtain the partial derivatives of g and since the Cauchy-Riemann equations hold for f

$$\begin{aligned} \frac{\partial s}{\partial x} &= x\frac{\partial u}{\partial x} + y\frac{\partial v}{\partial x} \\ \frac{\partial t}{\partial x} &= x\frac{\partial v}{\partial x} - y\frac{\partial u}{\partial x} \\ \frac{\partial s}{\partial y} &= -x\frac{\partial v}{\partial x} + y\frac{\partial u}{\partial x} \\ \frac{\partial t}{\partial y} &= x\frac{\partial u}{\partial x} + y\frac{\partial v}{\partial x}. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\partial s}{\partial x} &= \frac{\partial t}{\partial y} \\ \frac{\partial t}{\partial x} &= -\frac{\partial s}{\partial y}. \end{aligned}$$

Method 2, using the definition. Let $z \in \mathbb{C}$ and $f\colon \mathbb{C} \to \mathbb{C}$ entire. Now

$$\begin{aligned}
& g(z) = \bar{z} f(z) \text{ is differentiable} \\
& \quad \Leftrightarrow \quad \lim_{h \to 0} \frac{g(z+h) - g(z)}{h} = g'(z) \text{ exists} \\
& \quad \Leftrightarrow \quad g'(z) = \lim_{h \to 0} \frac{\overline{z+h} f(z+h) - \bar{z} f(z)}{h} \\
& \qquad = \lim_{h \to 0} \left(\bar{z} \left(\frac{f(z+h) - \bar{z} f(z)}{h} \right) + \frac{\bar{h} f(z+h)}{h} \right) \\
& \qquad = \bar{z} f'(z) + \lim_{h \to 0} \frac{\bar{h} f(z+h)}{h}.
\end{aligned}$$

If $\lim_{h \to 0} f(z+h) \neq 0$, then

$$\lim_{h \to 0} \frac{\bar{h} f(z+h)}{h} = \begin{cases} f(z+h) & \text{when } h \in \mathbb{R} \\ -f(z+h) & \text{when } h \in \mathbb{C} \setminus \mathbb{R}. \end{cases}$$

So $g'(z)$ does not exist if $\lim_{h \to 0} f(z+h) \neq 0$.

If $\lim_{h \to 0} f(z+h) = 0$, then $\lim_{h \to 0} \frac{\bar{h} f(z+h)}{h}$ exists and equals zero.

Since f is analytic, f is continuous and we have

$$\lim_{h \to 0} f(z+h) = 0 = f(z).$$

Exercise 2. Let A and B be open sets in $\mathbb{C}$. Let f be analytic on A and g be analytic on B and $f(A) \subseteq B$. Then, the composite function $g \circ f$, given by $(g \circ f)(z) = g(f(z))$, is analytic on A and for all $z \in A$

$$(g \circ f)'(z) = g'(f(z))f'(z).$$

Proof: Since f and g are complex differentiable in A and B, then for all $z \in A$ and $\omega \in B$ there exists $h \in \mathbb{C}$ and $k \in \mathbb{C}$ such that

$$f(z+h) - f(z) = f'(z)h + h\varepsilon_{f,z}(h) \text{ and } g(\omega + k) - g(\omega) = g'(\omega)k + k\varepsilon_{g,\omega}(k),$$

where $\varepsilon_{f,z}(h) \to 0$ as $h \to 0$ and $\varepsilon_{g,\omega}(k) \to 0$ as $k \to 0$.

Since $f(z) \in B$ for every $z \in A$, the g is complex differentiable at $f(z)$, that is,

$$g(f(z) + k) - g(f(z)) = g'(f(z))k + k\varepsilon_{g,f(z)}(k).$$

When we choose $k = f(z+h) - f(z)$, then

$$g(f(z+h)) - g(f(z)) = g'(f(z))(f(z+h) - f(z)) + (f(z+h) - f(z))\varepsilon_{g,f(z)}(k).$$

Since $f(z+h) - f(z) = f'(z)h + h\varepsilon_{f,z}(h)$, then

$$\begin{aligned} &g(f(z+h)) - g(f(z)) = g'(f(z))(f(z+h) - f(z)) + (f(z+h) - f(z))\varepsilon_{g,f(z)}(k) \\ =&g'(f(z))f'(z)h + g'(f(z))h\varepsilon_{f,z}(h) + (f'(z)h + h\varepsilon_{f,z}(h))\varepsilon_{g,f(z)}(k) \\ =&g'(f(z))f'(z)h + h(g'(f(z))\varepsilon_{f,z}(h) + f'(z)\varepsilon_{g,f(z)}(k) + \varepsilon_{f,z}(h)\varepsilon_{g,f(z)}(k). \end{aligned}$$

Now we can define function

$$\varepsilon_{g \circ f,z} = g'(f(z))\varepsilon_{f,z}(\delta) + f'(z)\varepsilon_{g,f(z)}(\delta) + \varepsilon_{f,z}(\delta)\varepsilon_{g,f(z)}(\delta),$$

where $\delta = \min\{h, k\}$. Then it holds

$$\varepsilon_{g \circ f,z}(\delta) \to 0 \text{ as } \delta \to 0.$$

Now we have

$$g(f(z) + k) - g(f(z)) = g'(f(z))f'(z) + \delta\varepsilon_{g \circ f,z}(\delta),$$

so $g \circ f$ is analytic on A and the derivative is $g'(f(z))f'(z)$.

Exercise 3. The function f is not complex differentiable at origin. We can show that by studying the limits. When $\operatorname{Im}(h) = 0$, then

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h} = 0.$$

But when $h = h_1 + ih_2$ where $h_1 = h_2^2$, then

$$\lim_{h\to 0} \frac{f(h^2+ih)}{h^2+ih} = \lim_{h\to 0} \frac{\frac{h^2h^2(h^2+ih)}{h^4+h^4}}{h^2+ih} = \lim_{h\to 0} \frac{\frac{h^4(h^2+ih)}{2h^4}}{h^2+ih} = \lim_{h\to 0} \frac{\frac{h^2+ih}{2}}{h^2+ih}$$
$$= \lim_{h\to 0} \frac{h^2+ih}{2(h^2+ih)} = \frac{1}{2}.$$

So f is not complex differentiable at origin. Then we find the points where f is complex differentiable. We define new function $g : \mathbb{C} \to \mathbb{C}$ such that $g(0) = 0$ and

$$g(x, y) = \frac{f(x, y)}{x + iy} = \frac{\frac{xy^2(x+iy)}{x^2+y^4}}{x + iy} = \frac{xy^2}{x^2 + y^2}.$$

We can see that g is real differentiable at every $z \neq 0$ by studying the partial derivatives:

$$\frac{\partial g}{\partial x} = \frac{y^2(x^2+y^4) - 2x^2y^2}{(x^2+y^4)^2} = \frac{y^2(y^4 - x^2)}{(x^2+y^4)^2}$$
$$\frac{\partial g}{\partial y} = \frac{2xy(x^2+y^4) - 4xy^5}{(x^2+y^4)^2} = \frac{2yx^3 + 2xy^5 - 4xy^5}{(x^2+y^4)^2} = \frac{2xy(x^2 - y^4)}{(x^2+y^4)^2}.$$

Now, g is complex differentiable at $(x_0, y_0) \in \mathbb{C} \setminus \{0\}$, <u>iff</u> Cauchy-Riemann equations are satiesfied, that is, if

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \text{ and } \frac{\partial v}{\partial x}(x_0, y_0) = -\frac{\partial u}{\partial y}(x_0, y_0).$$

When $g = u + iv$, then $u = g$ and $v = 0$. That is, we must find points where

$$\frac{\partial u}{\partial x}(x_0, y_0) = 0 \text{ and } \frac{\partial u}{\partial y}(x_0, y_0) = 0.$$

Now

$$\frac{\partial u}{\partial y}(x_0, y_0) = 0 \Rightarrow \frac{\partial g}{\partial y}(x_0, y_0) = 0 \Rightarrow \frac{2xy(x^2 - y^4)}{(x^2+y^4)^2} = 0 \Rightarrow 2xy(x^2 - y^4) = 0$$
$$\Rightarrow 2x^3y = 2xy^5 \Rightarrow x^2y = y^5 \Rightarrow |x| = |y|^4.$$

Calculating $\frac{\partial u}{\partial x}(x_0, y_0) = 0$ gives the same result. So g is differentiable on a set

$$\Omega = \{(x, y) \in \mathbb{C}\colon |x| = |y|^4\} \cup \{(x, 0) \in \mathbb{C}\colon x \neq 0\} \cup \{(0, y) \in \mathbb{C}\colon y \neq 0\}.$$

Since Ω is not open, then g is not analytic anywhere. Also, f is analytic on an open set if g is, therefore f is not analytic anywhere.

Exercise 4. Let $f : \mathbb{C} \to \mathbb{C}$, $f(z) = u(x,y) + iv(x,y)$ be entire and assume that:

$$2u(x,y) + v(x,y) = 5, \text{ for all } z = (x+iy) \in \mathbb{C}$$

From the given condition, we can solve $v(x,y)$ in terms of $u(x,y)$

$$v(x,y) = 5 - 2u(x,y)$$

and substitute this to the given expression of $f(z)$:

$$f(z) = u(x,y) + i(5 - 2u(x,y))$$

As f is entire, the Cauchy–Riemann equations must hold in the whole complex plane:

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \end{cases} \tag{2}$$

From expressing v in terms of u we now have the following expressions for the partial derivatives of v:

$$\frac{\partial v}{\partial y} = -2\frac{\partial u}{\partial y} \qquad \frac{\partial v}{\partial x} = -2\frac{\partial u}{\partial x}$$

Substituting these to the Cauchy-Riemann equations gives:

$$\begin{cases} \frac{\partial u}{\partial x} = -2\frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial y} = 2\frac{\partial u}{\partial x} \end{cases} \tag{3}$$

Which is a pair of linear equations with the partials of u as the two unknowns. The pair has the following unique solution:

$$\begin{cases} \frac{\partial u}{\partial x} = 0 \\ \frac{\partial u}{\partial y} = 0 \end{cases} \tag{4}$$

Now as both the partial derivatives of $u(x.y)$ are zero for all $x, y \in \mathbb{R}$, then $u(x,y)$ must be a constant function. As $v(x,y)$ is dependant on only $u(x,y)$ and constants, then $v(x,y)$ must also be constant. Finally as u and v are the real and imaginary parts of f, (and now are constant) f must also be a constant function. □

Exercise 5. Recall that, because function f is complex-valued, we may write its value

$$f(x+iy) = u(x,y) + iv(x,y)$$

for any complex number $(x+iy) \in \mathbb{D}$.

Suppose first that $\operatorname{Re} f$ is constant. Now $u(x,y) = \operatorname{Re} f$ is constant. Thus, partial derivatives of $u(x,y)$ have to satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0. \tag{5}$$

Because f is, by assumption, analytic in $\mathbb{D}$, it satisfies the Cauchy—Riemann equations

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \end{cases} \tag{6}$$

in $\mathbb{D}$. But substituting values from (5) into (6) gives

$$\begin{cases} 0 = \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} = -0, \end{cases}$$

and it follows that

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0.$$

All partial derivatives of function f are zero in $\mathbb{D}$. We conclude that f is constant in $\mathbb{D}$.

Suppose then that $\operatorname{Im} f$ is constant. Using notation from previous case, we have $v(x,y) = \operatorname{Im} f$. Partial derivatives of $v(x,y)$ now satisfy

$$\begin{cases} \frac{\partial v}{\partial x} = 0 \\ \frac{\partial v}{\partial y} = 0. \end{cases} \tag{7}$$

Because f is, by assumption, analytic in $\mathbb{D}$, it satisfies the Cauchy—Riemann equations (6) in $\mathbb{D}$. But substituting values from (7) into (6) gives

$$\begin{cases} \frac{\partial u}{\partial x} = 0 \\ 0 = -\frac{\partial u}{\partial y}, \end{cases}$$

and it follows that

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0.$$

All partial derivatives of function f are zero in $\mathbb{D}$. We conclude that f is constant in $\mathbb{D}$.

Suppose now that $|f|$ is constant. Then $|f|^2 = f\bar{f} = u^2(x,y) + v^2(x,y)$ is also constant, ie. its partial derivatives are zero in $\mathbb{D}$. It now suffices to show that all partial derivatives of f are also zero. Taking derivatives of $|f|^2$ with respect to x gives

$$\begin{aligned} & \frac{\partial}{\partial x}(u^2 + v^2) = 0 \\ \Rightarrow \quad & 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0 \\ \Rightarrow \quad & u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0. \end{aligned} \tag{8}$$

On the other hand, derivatives wrt. variable y yield

$$\begin{aligned} & \frac{\partial}{\partial y}(u^2 + v^2) = 0 \\ \Rightarrow \quad & 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0 \\ \Rightarrow \quad & u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0. \end{aligned} \tag{9}$$

Because f is analytic by assumption, it also satisfies the Cauchy—Riemann equations

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \end{cases}$$

in $\mathbb{D}$. Writing these together with (8) and (9) in matrix form gives

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ u & 0 & v & 0 \\ 0 & u & 0 & v \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

This is a system of homogeneous equations. Therefore, it always has the trivial solution (zero vector). But a unique solution must exist, because the columns of the coefficient matrix are linearly independent. This means that the unique solution is exactly the trivial solution

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Because all partial derivatives of analytic function f are zero in $\mathbb{D}$, we conclude that f is constant in $\mathbb{D}$.