Complex Analysis Homework 4

February 2020

4.1

Show that in polar coordinates, the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$$

From the polar expression of $z = x + iy = r(\cos\theta + ir\sin\theta)$ we know that: $x = r\cos\theta$ and $y = r\sin\theta$.
Applying the chain rule, and then applying the C-R equations:

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} =$$

$$\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta = \frac{1}{r}(r\frac{\partial v}{\partial y}\cos\theta - r\frac{\partial v}{\partial x}\sin\theta) =$$

$$\frac{1}{r}(\frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta}) = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

If we proceed in an analogous way we prove that

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}cos\theta + \frac{\partial v}{\partial y}\sin\theta = -\frac{1}{r}(r\frac{\partial u}{\partial y}\cos\theta - r\frac{\partial u}{\partial x}\sin\theta) = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

4.2

Define a function $f : \mathbb{C} \to \mathbb{C}$ by

$$f(x) = \begin{cases} \exp(-z^{-4}), & \text{if } z \neq 0, \\ 0, & \text{if } z = 0. \end{cases}$$

Show that the Cauchy-Riemann equations are valid at every point of the complex plane, but f is not an entire function.

The function f is analytic in $\mathbb{C} \setminus \{0\}$, since $z \mapsto \exp(z)$ is entire, and $z \mapsto \frac{1}{z^4}$ is analytic in $\mathbb{C} \setminus \{0\}$. We will then handle the remaining case $z = 0$ by using the definition of partial derivatives.

Recall that we may write $f(z) = f(x, y) = (u(x, y), iv(x, y))$. Now let $h \in \mathbb{R}$. Finding partial derivative $\frac{\partial f}{\partial x}$ gives, looking from positive x-axis,

$$\begin{aligned} \lim_{h \to 0+} \frac{f(0 + h(1, 0)) - f(0)}{h} &= \lim_{h \to 0+} \frac{\exp(-h^{-4}) - 0}{h} \\ &= \lim_{h \to 0+} \frac{\exp(-h^{-4})}{h}, \end{aligned}$$

which cannot be evaluated directly. By substituting $t = h^{-4}$, we get

$$\begin{aligned} \lim_{h \to 0+} \frac{\exp(-h^{-4})}{h} &= \lim_{t \to \infty} \frac{t^{\frac{1}{4}}}{e^t} && (1) \\ &= \lim_{t \to \infty} \frac{\frac{1}{4} t^{\frac{-3}{4}}}{e^t} && (2) \\ &= 0, \end{aligned}$$

where we used l'Hopital's rule to get from (1) to (2). A similar calculation for the negative x-axis gives

$$\lim_{h \to 0-} \frac{f(0 + h(1, 0)) - f(0)}{h} = 0.$$

Because derivatives from both directions exist and are zero, partial derivative $\frac{\partial f}{\partial x}(0)$ exists, and $\frac{\partial f}{\partial x}(0) = \frac{\partial u}{\partial x}(0) + \frac{\partial v}{\partial x}(0) = 0$.

In the same way, one can show that $\frac{\partial f}{\partial y}(0)$ exists, and that $\frac{\partial f}{\partial y}(0) = 0$. To conclude, we have

$$\frac{\partial u}{\partial x}(0) = \frac{\partial v}{\partial x}(0) = \frac{\partial u}{\partial y}(0) = \frac{\partial v}{\partial y}(0) = 0.$$

This means that component functions u and v satisfy the Cauchy—Riemann equations at the origin. But function f still fails to be entire: the following example shows that function f is not even continuous at $z = 0$.
Let $z = r \exp(\frac{i\pi}{4})$. Then $z^4 = r^4 \exp(i\pi)$ and

$$\lim_{r \to 0+} f\left(r \exp\left(\frac{i\pi}{4}\right)\right) = \lim_{r \to 0+} e^{\frac{1}{r^4}} = \infty.$$

4.3

For each of the following power series, calculate the radius of convergence.

$$(a) \quad \sum_{n=0}^{\infty} z^n, \qquad (b) \quad \sum_{n=1}^{\infty} nz^{n-1}, \qquad (c) \quad \sum_{n=1}^{\infty} n^2 z^{n-1}.$$

Find the corresponding sums.

(a) This is a geometric series: the radius of convergence is 1. Series converges when $|z| < 1$ and diverges when $|z| \geq 1$. The sum of the series is:

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}.$$

(b) This is the derivative of the first one. $\mathrm{D}\sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} nz^{n-1}$ The radius of convergence is 1. Series converges when $|z| < 1$ and diverges when $|z| \geq 1$. The sum of the series is:

$$\sum_{n=1}^{\infty} nz^{n-1} = \frac{d}{dz}\left(\frac{1}{1-z}\right) = \frac{1}{(1-z)^2}.$$

(c) Recall that differentiating a power series does not affect its radius of convergence (theorem 5.2.10 in the lecture notes). Noting that

$$\frac{d^2}{dz}\sum_{n=1}^{\infty} z^{n+1} = \sum_{n=1}^{\infty} n^2 z^{n-1} + nz^{n-1}$$

allows us to write the original series (c) in terms of series (a):

$$\begin{aligned} \sum_{n=1}^{\infty} n^2 z^{n-1} &= \sum_{n=1}^{\infty} n^2 z^{n-1} + nz^{n-1} - nz^{n-1} \\ &= \frac{d^2}{dz}\left[\sum_{n=1}^{\infty} z^{n+1}\right] - \sum_{n=1}^{\infty} nz^{n-1} \\ &= \frac{d^2}{dz}\left[\left(\sum_{n=1}^{\infty} z^n\right) \cdot z\right] - \frac{d}{dz}\left[\sum_{n=1}^{\infty} z^n\right]. \end{aligned} \tag{3}$$

Based on the solution of item (a) of this exercise, we know that, for $|z| < 1$,

$$\sum_{n=1}^{\infty} z^n = \frac{1}{1-z}.$$

Substituting this into (3) gives

$$\begin{aligned} \sum_{n=1}^{\infty} n^2 z^{n-1} &= \frac{d^2}{dz}\left[\left(\frac{1}{1-z}\right) \cdot z\right] - \frac{d}{dz}\left[\frac{1}{1-z}\right] \\ &= \frac{2z}{(1-z)^3} + \frac{1}{(1-z)^2} = \frac{1+z}{(1-z)^3}, \end{aligned}$$

which is the required sum of series (c). Furthermore, because we have shown that series (c) consists of derivatives of series (a), it has the same radius of convergence as series (a). Thus, series (c) converges when $|z| < 1$.

4.4

(1) Suppose that $\sum a_n z^n$ has the radius of convergence R. Find the radius of convergence of $\sum a_n z^{2n}$.
(2) Discuss the convergence of the series $\sum_{n=0}^{\infty} 2^n |z|^{n^2}$.

(1) By Hadamard's theorem, the radius of convergence of $\sum a_n z^n$ is given by the formula

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} \Leftrightarrow R = \frac{1}{\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}}.$$

Define

$$b_k = \begin{cases} a_{\frac{k}{2}} & k \in 2\mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

We have k = 2n for all n $\in \mathbb{N}$ and $\sum b_k z^k = \sum a_n z^{2n}$. By Hadamard's theorem,

$$\frac{1}{\limsup_{k\to\infty} |b_k|^{\frac{1}{k}}} = \frac{1}{\limsup_{k\to\infty} |a_{\frac{k}{2}}|^{\frac{1}{k}}} = \frac{1}{\limsup_{n\to\infty} |a_n|^{\frac{1}{2n}}} = \left(\frac{1}{\limsup_{k\to\infty} |a_n|^{\frac{1}{n}}}\right)^{\frac{1}{2}} = \sqrt{R},$$

hence the radius of convergence of $\sum a_n z^{2n}$ is $\sqrt{R}$.

(2) The terms of the series $\sum_{n=0}^{\infty} 2^n |z|^{n^2}$ are non-negative, so we may use the root test.

$$\lim_{n\to\infty} \sqrt[n]{|2^n |z|^{n^2}|} = \lim_{n\to\infty} 2|z|^n = \begin{cases} 0 & |z| < 1 \\ 2 & |z| = 1 \\ \infty & |z| > 1 \end{cases}$$

Therefore the series converges absolutely when $|z| < 1$ and diverges otherwise.

4.5

Let (a_n) be a sequence of complex numbers such that $\sum |a_n| < \infty$ and $\sum n|a_n| = \infty$. Prove that the radius of convergence of $\sum a_n z^n$ is 1

We will prove that $R = 1$. Because $\sum |a_n| < \infty$, we get

$$\sum a_n z^n \leq \sum |a_n| z^n \leq \sum |a_n| < \infty,$$

when $|z| \leq 1$. Thus $R >= 1$.

For $R <= 1$ we use proof by contradiction. Suppose $R > 1$. Now $\frac{1}{R} < 1$. Hadamard's theorem tells that

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}.$$

Because this is less than one, we can find such $N \in \mathbb{Z}$ and $\epsilon > 0$ that $|a_n|^{\frac{1}{n}} < 1-\epsilon$ for all $n > N$. Thus

$$|a_n| < (1-\epsilon)^n$$

for $n > N$.

We have $\sum na_n < \infty$ if $\sum_{n=k}^{\infty} na_n < \infty$. We also know that

$$\sum \frac{1}{n^l} < \infty$$

iff $l > 1$. Thus $\sum_{n=k}^{\infty} na_n < \infty$, if we prove that for all sufficiently large n and some $l > 1$

$$na_n < \frac{1}{n^l}$$

holds. That can be written as

$$a_n < \frac{1}{n^{l+1}}.$$

Lets pick $l = 2$. We want $a_n < \frac{1}{n^3}$ for sufficiently large n. We know that $a_n < (1-\epsilon)^n$ for sufficiently large n, so we want $(1-\epsilon)^n < \frac{1}{n^3}$.

$$(1-\epsilon)^n < \frac{1}{n^3}$$

$$n^3 < \left(\frac{1}{1-\epsilon}\right)^n.$$

Since $\frac{1}{1-\epsilon} > 1$, this holds for sufficiently large n. Now we have a contradiction with the assumption $\sum n|a_n| = \infty$, so $R \leq 1$. Thus $R = 1$.