Complex Analysis Homework 5 Solutions

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Exercise 1. If we choose a complex number that has a real part $1 + a\cos\alpha + a^2\cos 2\alpha ...$ and imaginary part $a\sin\alpha + a^2\sin 2\alpha + a^3\sin 3\alpha ...$, we get:

$$\begin{aligned}(1 + a\cos\alpha + a^2\cos 2\alpha + a^3\cos 3\alpha ...) + i(a\sin\alpha + a^2\sin 2\alpha + a^3\sin 3\alpha ...)\\ = (a^0\cos 0\alpha + a^1\cos 1\alpha + a^2\cos 2\alpha + a^3\cos 3\alpha ...)\\ + i(a^0\sin 0\alpha + a\sin\alpha + a^2\sin 2\alpha + a^3\sin 3\alpha ...)\end{aligned}$$

We can write this as:

$$\begin{aligned}\sum_{n=0}^{\infty} a^n e^{in\alpha} &= \sum_{n=0}^{\infty}(ae^{i\alpha})^n\\ &= \frac{1}{1-ae^{i\alpha}}\\ &= \frac{1-ae^{-i\alpha}}{(1-e^{i\alpha})(1-e^{-i\alpha})}\\ &= \frac{1-ae^{-i\alpha}}{1-ae^{i\alpha}-ae^{-i\alpha}+a^2}\\ &= \frac{1-ae^{-i\alpha}}{1-a(e^{i\alpha}+e^{-i\alpha})+a^2}\\ &= \frac{1-a(\cos\alpha - i\sin\alpha)}{1-a(\cos\alpha + i\sin\alpha + \cos\alpha - i\sin\alpha)+a^2}\\ &= \frac{(1-a\cos\alpha)+(ai\sin\alpha)}{1-2a\cos\alpha+a^2}\\ &= \frac{1-a\cos\alpha}{1-2a\cos\alpha+a^2} + i\frac{a\sin\alpha}{1-2a\cos\alpha+a^2}\end{aligned}$$

Since we assigned $1 + a\cos\alpha + a^2\cos 2\alpha ...$ to be the real part, we can see that it equals to $\frac{1-a\cos\alpha}{1-2a\cos\alpha+a^2}$. In the same way we see that the imaginary part $a\sin\alpha + a^2\sin 2\alpha + a^3\sin 3\alpha ...$ results in $\frac{a\sin\alpha}{1-2a\cos\alpha+a^2}$.

Exercise 2. (1) By Euler's formula we know that for every $z = x + iy \in \mathbb{C}$

$$\exp(z) = \exp(x+iy) = e^x(\cos(y) + i\sin(y)).$$

Using this formula twice we get

$$\begin{aligned}\exp(\exp(z)) &= \exp(e^x(\cos(y) + i\sin(y))) \\ &= e^{e^x \cos(y)}(\cos(e^x \sin(y)) + i\sin(e^x \sin(y))).\end{aligned}$$

Therefore

$$\operatorname{Re}(z) = e^{e^x \cos(y)} \cos(e^x \sin(y)) \text{ and } \operatorname{Im}(z) = e^{e^x \cos(y)} \sin(e^x \sin(y)).$$

(2) Notice that $\exp(Cz)$ is entire and

$$\frac{\partial}{\partial z}\exp(Cz) = C\exp(Cz).$$

When f is entire and $f'(z) = Cf(z)$, then

$$\begin{aligned}\frac{\partial}{\partial z}\frac{f(z)}{\exp(Cz)} &= \frac{C\exp(Cz)f(z) - \exp(Cz)f'(z)}{(\exp(Cz))^2} \\ &= \frac{C\exp(Cz)f(z) - C\exp(Cz)f(z)}{(\exp(Cz))^2} = 0.\end{aligned}$$

That is,

$$\frac{f(z)}{\exp(Cz)} = C'$$

where C' is constant. From this we get

$$f(z) = C'\exp(Cz).$$

So all the functions f that satisfy the given property can be written as $f(z) = C'\exp(Cz)$.

Exercise 3. Suppose that $z, z_1, z_2 \in \mathbb{C}$ and $x, y \in \mathbb{R}$. First, we will show that

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2.$$

First, we calculate the product $\sin z_1 \cos z_2$ using the definitions of complex sine and cosine:

$$\begin{aligned}\sin z_1 \cos z_2 &= \frac{1}{2i \cdot 2}(\exp(iz_1) - \exp(-z_1)(\exp(iz_2) + \exp(-iz_2)) \\ &= \frac{1}{4i}(\exp(iz_1)\exp(iz_2) + \exp(iz_1)\exp(-iz_2) \\ &\qquad - \exp(-iz_1)\exp(iz_2) - \exp(-iz_1)\exp(-iz_2)).\end{aligned}$$

We notice that $\cos z_1 \sin z_2 = \sin z_2 \cos z_1$. In this way we can use the same calculation as above (change z_1 to z_2 and vv.) and obtain:

$$\begin{aligned}\sin z_2 \cos z_1 = \frac{1}{4i}(&\exp(iz_2)\exp(iz_1) + \exp(iz_2)\exp(-iz_1) \\ &- \exp(-iz_2)\exp(iz_1) - \exp(-iz_2)\exp(-iz_1)).\end{aligned}$$

By comparing these two products and taking their sum, we obtain:

$$\begin{aligned}
\sin z_1 \cos z_2 + \cos z_1 \sin z_2 &= \frac{2\exp(iz_1)\exp(iz_2) - 2\exp(-iz_1)\exp(-iz_2)}{4i} \\
&= \frac{\exp(iz_1)\exp(iz_2) - \exp(-iz_1)\exp(-iz_2)}{2i} \\
&= \frac{\exp(iz_1 + iz_2) - \exp(-iz_1 - iz_2)}{2i} \\
&= \frac{\exp(i(z_1 + z_2)) - \exp(-i(z_1 + z_2))}{2i} \\
&= \sin(z_1 + z_2).
\end{aligned}$$

Secondly, we will prove Osborn's rules:

$$\sin iz = i \sinh z \text{ and } \cos iz = \cosh z.$$

By using the definitions of complex sine and hyperbolic sine we obtain:

$$\begin{aligned}
\sin iz &= \frac{1}{2i}(\exp(iiz) - \exp(-iiz)) \\
&= \frac{1}{2i}(\exp(-z) - \exp(z)) \\
&= \frac{-i}{2}(\exp(-z) - \exp(z)) \\
&= \frac{i}{2}(\exp(z) - \exp(-z)) \\
&= i \sinh z.
\end{aligned}$$

For complex cosine we have:

$$\begin{aligned}
\cos iz &= \frac{1}{2}(\exp(iiz) + \exp(-iiz)) \\
&= \frac{1}{2}(\exp(-z) + \exp(z)) \\
&= \cosh z.
\end{aligned}$$

Lastly, we will show that

$$\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y.$$

To prove this we will apply results that we already proved in this exercise. The first formula that we proved, gives us

$$\sin(x + iy) = \sin x \cos iy + \cos x \sin iy.$$

To this we apply Osbourn's rules, which finally gives us

$$\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y.$$

Exercise 4. Let $z \in \mathbb{C}$ and $\sin z = -2i$. Using the definition of complex sine this can be written as

$$\frac{1}{2i}(\exp(iz) - \exp(-iz)) = -2i$$

which with substitution of $w = \exp(iz)$ and multiplication by $2i$ becomes:

$$w - \frac{1}{w} = 4 \Longleftrightarrow w^2 - 4w - 1 = 0$$

This can be solved with the quadratic formula, from which we get two roots for the equation:

$$w = \frac{-(-4) \pm \sqrt{16 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1} = \frac{4 \pm 2\sqrt{5}}{2} = 2 \pm \sqrt{5}$$

In general when $z, w \in \mathbb{C}$, for the solution of the equation $\exp(iz) = w$ we have:

$$\begin{aligned} iz &= \ln|w| + i \arg w \\ \Longleftrightarrow z &= -i \ln|w| + \arg w \end{aligned}$$

Using this information we will now consider the first root $w_1 = 2 + \sqrt{5}$. This is a positive, purely real number and is therefore positioned on the positive real axis with an argument of $2\pi k$ in the complex plane with $k \in \mathbb{Z}$. With the formula derived above we hence get:

$$z_1 = -i \ln|2 + \sqrt{5}| + \arg(2 + \sqrt{5}) = -i \ln(2 + \sqrt{5}) + 2\pi k$$

For the other root $w_2 = 2 - \sqrt{5}$ we note that it is is purely real and negative. Therefore it is positioned on the negative real axis and the argument will be $\pi + 2\pi k$ with $k \in \mathbb{Z}$. Using the knowledge $w_2 < 0$ we can also get rid of the absolute value:

$$z_2 = -i \ln|2 - \sqrt{5}| + \arg(2 - \sqrt{5}) = -i \ln(-2 + \sqrt{5}) + \pi + 2\pi k$$

Hence values (where $k \in \mathbb{Z}$) that solve the equation $\sin z = -2i$ will be

$$z = -i \ln(2 + \sqrt{5}) + 2\pi k$$

or

$$z = -i \ln(-2 + \sqrt{5}) + \pi + 2\pi k$$

Exercise 5. To find the branch of the logarithm and the i^{th} power of $(-\sqrt{3} - i)$ we first determine the arguments and the modulus from the basic trigonometric triangles:

$$\begin{aligned} &\mathrm{Arg}_{(-\pi/2, 3\pi/2]}(-\sqrt{3} - i) = 7\pi/6 \\ &\mathrm{Arg}_{(-\pi, \pi]}(-\sqrt{3} - i) = -5\pi/6 \\ &|z| = 2 \end{aligned}$$

(1) Now by the definition of complex logarithm:

$$\mathrm{Log}_{(-\pi/2,3\pi/2]}(-\sqrt{3}-i) = \ln|z| + i\,\mathrm{Arg}_{(-\pi/2,3\pi/2]}(-\sqrt{3}-i) = \ln 2 + i7\pi/6$$
$$(-\sqrt{3}-i)^i = \exp(i\,\mathrm{Log}_{(-\pi/2,3\pi/2]}(-\sqrt{3}-i))$$
$$= \exp(i\ln 2)\exp(i(i7\pi/6)) = e^{-7\pi/6}(\cos(\ln 2) + i\sin(\ln 2))$$

(2) Similarly:

$$\mathrm{Log}_{(-\pi,\pi]}(-\sqrt{3}-i) = \ln 2 - i5\pi/6$$
$$(-\sqrt{3}-i)^i = e^{5\pi/6}(\cos(\ln 2) + i\sin(\ln 2))$$

(3)

$$i^{\sqrt{2}} = \exp(\sqrt{2}\log(1)) = \exp(\sqrt{2}(\ln 1 + i2\pi k)) = \exp(i2\sqrt{2}\pi k)$$

$$i^{-2i} = \exp(-2i\log(i)) = \exp(-2i(i\pi/2 + i2\pi k))$$
$$= \exp(\pi + 4\pi k) = e^{\pi+4\pi k}$$
$$\implies \mathrm{Re}(i^{-2i}) = e^{\pi+4\pi k}$$

$$(-i)^{(-i)} = \exp(-i\log(i)) = \exp(-i(-i\pi/2 + i2\pi k))$$
$$= \exp(-\pi/2 + 2\pi k)$$
$$\implies |(-i)^{(-i)}| = e^{-\pi/2+2\pi k}$$

(4) First notice that

$$z^{1-z} = \exp((1-i)(\ln|z| + i\,\mathrm{Arg}_{(\alpha,\alpha+2\pi]}\,z).$$

Since we wish to satisfy

$$i^{1-z} = \exp(-3\pi/2 - i3\pi/2),$$

choose $\alpha = -3\pi$, now:

$$i^{1-z} = \exp((1-i)(\ln|i| + i\,\mathrm{Arg}_{(-3\pi,-\pi]}\,i))$$
$$= \exp((1-i)i(-3\pi/2)) = \exp((1+i)(-3\pi/2))$$
$$= \exp(-3\pi/2 - 3\pi/2),$$

which is what we wanted the equation to satisfy.