Complex Analysis Homework 6 Solutions

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Exercise 1. (1) Let $a, b \in \mathbb{C}$ and define $\gamma\colon [0,1] \to \mathbb{C}$ by $\gamma(t) = (1-t)a + tb$. We will evaluate the integral $\int_\gamma z \,\mathrm{d}z$. The function $f\colon \mathbb{C} \to \mathbb{C}$, $f(z) = z$ is continuous and γ is a smooth curve. For γ we have

$$\gamma'(t) = -a + b = b - a \text{ for all } t \in [0,1].$$

Now,

$$\begin{aligned}
\int_\gamma f(z)\,\mathrm{d}z &= \int_\gamma z\,\mathrm{d}z \\
&= \int_0^1 ((1-t)a + tb)(b-a)\,\mathrm{d}t \\
&= \int_0^1 (a - at + bt)(b-a)\,\mathrm{d}t \\
&= \int_0^1 a(b-a) - a(b-a)t + b(b-a)t\,\mathrm{d}t \\
&= \Big/_0^1 a(b-a)t - \frac{a(b-a)}{2}t^2 + \frac{b(b-a)}{2}t^2 \\
&= a(b-a) - \frac{a(b-a)}{2} + \frac{b(b-a)}{2} \\
&= \frac{2ab - 2a^2 + b^2 - 2ab + a^2}{2} \\
&= \frac{b^2 - a^2}{2}.
\end{aligned}$$

(2) Let $b \in \mathbb{C}$ and let r, α be positive real numbers. We define $\gamma\colon [0,\alpha] \to \mathbb{C}$ by $\gamma(t) = b + r\exp(it)$. Now let $f\colon \mathbb{C} \setminus \{b\} \to \mathbb{C}$ be defined by $f(z) = \frac{1}{z-b}$. Now f is continuous and well defined in its domain and γ is smooth. We

have $\gamma'(t) = ir\exp(it)$ for all $t \in [0, \alpha]$. Thus

$$\begin{aligned}
\int_\gamma f(z)\,\mathrm{d}z &= \int_0^\alpha f(\gamma(t))\gamma'(t)\,\mathrm{d}t \\
&= \int_0^\alpha \frac{ir\exp(it)}{r\exp(it)}\,\mathrm{d}t \\
&= \int_0^\alpha i\,\mathrm{d}t = \Big/_{0}^{\alpha}\; it \\
&= i\alpha - i0 \\
&= i\alpha.
\end{aligned}$$

(3) We define $\gamma\colon [0, 2\pi] \to \mathbb{C}$ by $\gamma(t) = b + r\exp(it)$. Let $n \in \mathbb{Z}$ and $f\colon \mathbb{C} \setminus \{b\} \to \mathbb{C}$ be defined by $f(z) = (z-b)^n$. We know that f is continuous everywhere in its domain for all possible values of n. The curve γ is smooth with derivative $\gamma'(t) = ir\exp(it)$ for all $t \in [0, 2\pi]$. We will evaluate the integral

$$\int_\gamma (z-b)^n\,\mathrm{d}z.$$

If $n \neq -1$, we have

$$\begin{aligned}
\int_\gamma (z-b)^n\,\mathrm{d}z &= \int_0^{2\pi} ir^{n+1}\exp((n+1)it)\,\mathrm{d}t \\
&= ir^{n+1}\int_0^{2\pi} \exp((n+1)it)\,\mathrm{d}t \\
&= \frac{ir^{n+1}}{(n+1)i}\;\Big/_{0}^{2\pi}\; \exp((n+1)it) \\
&= \frac{ir^{n+1}}{(n+1)i}(\exp((n+1)2\pi i) - \exp(0)) \\
&= \frac{ir^{n+1}}{(n+1)i}(1-1) = 0.
\end{aligned}$$

If $n = -1$, we can use the result in (2):

$$\int_\gamma (z-b)^{-1}\,\mathrm{d}z = 2\pi i.$$

Exercise 2. Let $\gamma_m(t) = t + it^m$, $t \in [0, 1]$, where $m \in \mathbb{N}$ is fixed. Evaluate the following integrals

$$\int_{\gamma_m} z\,dz \quad \text{and} \quad \int_{\gamma_m} \bar{z}\,dz.$$

Let's first note that functions $z \mapsto z$ and $z \mapsto \bar{z}$ are continuous in $\mathbb{C}$ and γ_m is a smooth curve for any $m \in \mathbb{N}$, $\gamma_m'(t) = 1 + imt^{m-1}$. Thus, we can use the definition for integrating along a curve,

$$\int_{\gamma_m} f(z)\,dz = \int_0^1 f(\gamma_m(t))\gamma_m'(t)\,dt.$$

First integral:

$$\begin{aligned}
\int_{\gamma_m} z\,dz &= \int_0^1 (t+it^m)(1+imt^{m-1})\,dt \\
&= \int_0^1 \Big(t + imt^m + it^m - mt^{2m-1}\Big)\,dt \\
&= \int_0^1 \Big(t - mt^{2m-1} + i(m+1)t^m\Big)\,dt \\
&= \Big/_0^1 \Big(\frac{1}{2}t^2 - \frac{1}{2}t^{2m} + it^{m+1}\Big) \\
&= \frac{1}{2} - \frac{1}{2} + i \\
&= i
\end{aligned}$$

Second integral:

$$\begin{aligned}
\int_{\gamma_m} z\,dz &= \int_0^1 (t-it^m)(1+imt^{m-1})\,dt \\
&= \int_0^1 \Big(t + imt^m - it^m + mt^{2m-1}\Big)\,dt \\
&= \int_0^1 \Big(t + mt^{2m-1} + i(m-1)t^m\Big)\,dt \\
&= \Big/_0^1 \Big(\frac{1}{2}t^2 + \frac{1}{2}t^{2m} + i\frac{m-1}{m+1}t^{m+1}\Big) \\
&= \frac{1}{2} + \frac{1}{2} + i\frac{m-1}{m+1} \\
&= 1 + i\frac{m-1}{m+1}
\end{aligned}$$

Exercise 3. Define $\gamma\colon [0,\pi] \to \mathbb{C}$, $\gamma(t) = R\exp(it)$ where $R > 3$. Show that

$$\Big|\int_\gamma \frac{\exp(3iz)}{(z^2+4)(z^2+9)}\,\mathrm{d}z\Big| \le \frac{\pi R}{(R^2-4)(R^2-9)}.$$

Solution

First estimate $|\exp(3iz)|$. By Euler's formula

$$\begin{aligned}\big|\exp(3iz)\big| &= \big|\exp(3iR\exp(it))\big| \\ &= \big|\exp(3iR(\cos(t)+i\sin(t))\big| \\ &= \big|\exp(3iR\cos(t)-3R\sin(t))\big| \\ &= \left|\frac{\exp(3iR\cos(t))}{\exp(3R\sin(t))}\right| \\ &= \frac{1}{e^{3R\sin(t)}} \le \frac{1}{e^0} = 1,\end{aligned}$$

since $t \in [0,\pi]$ and therefore $\sin(t) \ge 0$.

Furthermore we have $|\gamma(t)| = |R\exp(it)| = R$. Using this and the triangle inequality, we obtain

$$\begin{aligned}\big|\gamma^2(t)\big| &= \big|\gamma^2(t)+4-4\big| \le \big|\gamma^2(t)+4\big| + 4 \\ \Rightarrow \big|\gamma^2(t)+4\big| &\ge \big|\gamma^2(t)\big| - 4 = R^2 - 4.\end{aligned}$$

Proceeding similarly, we obtain

$$\big|\gamma^2(t)+9\big| \ge R^2 - 9.$$

To use the estimation lemma we need to calculate the length(γ):

$$\begin{aligned}\text{length}(\gamma) &= \int_0^\pi \big|\gamma'(t)\big|\,\mathrm{d}t \\ &= \int_0^\pi \big|Ri\exp(it)\big|\,\mathrm{d}t \\ &= \int_0^\pi \big|Ri\big|\big|\exp(it)\big|\,\mathrm{d}t \\ &= R\int_0^\pi 1\,\mathrm{d}t = R\pi\end{aligned}$$

Now by the estimation lemma

$$\begin{aligned}\Big|\int_\gamma \frac{\exp(3iz)}{(z^2+4)(z^2+9)^2}\,\mathrm{d}z\Big| &\le \sup_{t\in[0,\pi]} \Big|\frac{\exp(3i\gamma\,(t))}{(\gamma^2\,(t)+4)(\gamma^2\,(t)+9)}\Big|\,\text{length}(\gamma) \\ &\le \frac{1}{(R^2-4)(R^2-9)}\pi R = \frac{\pi R}{(R^2-4)(R^2-9)}.\end{aligned}$$

Exercise 4. γ is the contour that goes vertically from 0 to i, and then horizontally from i to i + 1. We can express it as contours γ_1 and γ_2, where:

$$\begin{aligned}
\gamma_1 &: [0,1] \to \mathbb{C} \\
\gamma_1(t) &= it \\
\gamma_1'(t) &= i \\
\gamma_2 &: [0,1] \to \mathbb{C} \\
\gamma_2(t) &= t + i \\
\gamma_2'(t) &= 1
\end{aligned}$$

$$[0,1] \to \mathbb{C}$$

$$\begin{aligned}
\int_\gamma |z|^2 \,\mathrm{d}t &= \int_0^1 |\gamma_1(t)|^2 \gamma_1'(t)\,\mathrm{d}t + \int_0^1 |\gamma_2(t)|^2 \gamma_2'(t)\,\mathrm{d}t \\
&= \int_0^1 |it|^2 i\,\mathrm{d}t + \int_0^1 |t+i|^2\,\mathrm{d}t \\
&= \int_0^1 t^2 i\,\mathrm{d}t + \int_0^1 t^2 + 1\,\mathrm{d}t \\
&= \Big/_0^1 \frac{i}{3}t^3 + \Big/_0^1 \frac{1}{3}t^3 + t \\
&= \frac{i}{3} + \frac{1}{3} + 1 \\
&= \frac{4}{3} + \frac{i}{3}
\end{aligned}$$

Then we do this by going first from 0 to 1, and then to i + 1:

$$\begin{aligned}
\gamma_3 &: [0,1] \to \mathbb{C} \\
\gamma_3(t) &= t \\
\gamma_3'(t) &= 1 \\
\gamma_4 &: [0,1] \to \mathbb{C} \\
\gamma_4(t) &= it + 1 \\
\gamma_4'(t) &= i
\end{aligned}$$

$$
\begin{aligned}
\int_\gamma |z|^2 \,\mathrm{d}t &= \int_0^1 \left|\gamma_3(t)\right|^2 \gamma_3'(t) \,\mathrm{d}t + \int_0^1 \left|\gamma_4(t)\right|^2 \gamma_4'(t) \,\mathrm{d}t \\
&= \int_0^1 \left|t\right|^2 \,\mathrm{d}t + \int_0^1 \left|it+1\right|^2 i \,\mathrm{d}t \\
&= \int_0^1 t^2 \,\mathrm{d}t + \int_0^1 it^2 + i \,\mathrm{d}t \\
&= \Big/_0^1 \frac{1}{3}t^3 + \Big/_0^1 \frac{i}{3}t^3 + it \\
&= \frac{1}{3} + \frac{i}{3} + i \\
&= \frac{1}{3} + \frac{4i}{3}
\end{aligned}
$$

These two contours form a closed contour forming a square-shaped path between points 0, i, i+1, 1. If the mapping $z \mapsto |z|^2$ had an antiderivative, the integral on a closed path would equal to 0. Since

$$
\left(\frac{1}{3} + \frac{4i}{3}\right) - \left(\frac{4}{3} + \frac{i}{3}\right) = -1 + i \neq 0,
$$

the mapping does not have an antiderivative.

Exercise 5. Suppose that f is analytic in the unit disc $\mathbb{D}$, and f' is continuous there. If $\mathrm{Re}(f'(z)) > 0$ for all $z \in \mathbb{D}$, prove that f is one-to-one, that is, f is injective.

Solution.

We know that the disc $\mathbb{D}$ is convex, so for every pair of points $a, b \in \mathbb{D}$ we can find a path γ by $\gamma\colon [0,1] \to \mathbb{C}$, $\gamma(t) = bt + (1-t)a$. Now we want to show that if $a \neq b$ then $f(a) \neq f(b)$. We will show this by showing that $|f(b) - f(a)| > 0$. Now assume $a \neq b$ and since f' is continuous in $\mathbb{D}$

$$\begin{aligned}
|f(b) - f(a)| &= \left| \int_\gamma f'(z)dz \right| \\
&= \left| \int_a^b f'(\gamma(t))\gamma'(t)dt \right| \\
&= |b-a| \left| \int_a^b f'(bt + (1-t)a)dt \right| \\
&= |b-a| \left| \int_a^b \mathrm{Re}(f'(bt+(1-t)a))dt + i\int_a^b \mathrm{Im}(f'(bt+(1-t)a))dt \right| \\
&= \underbrace{|b-a|}_{>0} \left(\left(\int_a^b \underbrace{\mathrm{Re}(f'(bt+(1-t)a))}_{>0} dt \right)^2 + \left(\int_a^b \mathrm{Im}(f'(bt+(1-t)a))dt \right)^2 \right)^{\frac{1}{2}} > 0
\end{aligned}$$

this gives $f(a) \neq f(b)$. We have now shown that f is injective.